Lecture 4:

2. Nucleon Structure – continued

- Elementary Particles, Fundamental Interactions and the Standard Model
- Form factors

Textbook: Wong, Chapter 2
QED vs QCD

QED

DOES NOT EXIST

Photons interact only with charged particles

QCD

DOES EXIST

Gluons interact with color-charged particles (quarks and gluons)
The gluon in the strong force
Confinement

The photon does not carry electric charge.

$\alpha_{em} = \frac{e^2}{4\pi} \sim \frac{1}{137}$

Coupling constant: numerical coefficient that occur as a parameter whenever there’s an interaction.

Strength of interaction ~ magnitude of a coupling constant
The gluon in the strong force

The photon does not carry electric charge.

The gluon carries color charge itself.

\[ \alpha_{em} = \frac{e^2}{4\pi} \approx 1/137 \]

\[ \sqrt{\alpha} \sim 1 \text{ (large!)} \]

Solution: running $\alpha_s$

\[ \alpha \rightarrow \alpha_s (Q^2) \]

Extra diagrams depend on energy:
QCD at 10GeV = QCD at 1GeV, but with smaller coupling constant
Running coupling and asymptotic freedom in QCD

\[ \alpha_s(Q^2) = \frac{1}{33 - 2n_f} \times \frac{Q^2}{12\pi} \times \ln \frac{Q^2}{\Lambda^2} \]

where: \( n_f \) = number of quarks with mass < Q and \( \Lambda \sim 230 \text{ MeV} \)

The couplings, which set the strength for the interactions, change their value if one probes smaller distances with higher energies.
Running coupling and asymptotic freedom in QCD

\[ \alpha_s(Q^2) = \frac{1}{33 - 2n_f} \times \ln \frac{Q^2}{\Lambda^2} \]

\[ \Lambda^2 = \mu^2 \exp \left( \frac{-1}{B\alpha_s(\mu^2)} \right) \]

\[ \Lambda \sim 230 \text{ MeV} \]

The effective strong coupling decreases with energy (typical for non-Abelian fields, self-coupling gluons)

\[ Q^2 \rightarrow \alpha \quad \alpha_s \rightarrow 0 \quad \text{asymptotic freedom} \]

\[ \alpha_{em}(Q^2) = \frac{\alpha(\mu)}{\left[ 1 - \frac{1}{\pi} \alpha(\mu) \ln \left( \frac{Q^2}{\mu^2} \right) \right]} \]

\[ \alpha(\mu = 1 \text{ MeV}) = 1/137 \quad \text{(atomic physics)} \]

\[ \alpha(M_Z = 90 \text{ GeV}) = 1/129 \quad \text{(LEP e+e- accelerator)} \]

The effective em coupling increases with energy (or decreasing with smaller \( Q^2 \))

\[ \alpha_{em}(Q_1^2) < \alpha_{em}(Q_2^2) \quad Q_1^2 < Q_2^2 \]
Confinement: a crucial feature of QCD
(but no rigorous theoretical proof exists)

We can extract an electron from an atom by providing energy

But we cannot get free quarks out of hadrons: “colour confinement”

E = mc²

“white” proton (confined quarks)

“white” $\pi^0$ (confined quarks)

C. Lourenco (CERN)
Where to look for free quarks and gluons?

A very very long time ago... Quarks and gluons were “free”. As the universe cooled down, they got confined into hadrons.

Solution:
Recreate Big Bang conditions in the lab

Big Bang

Relativistic Heavy Ion Collider RHIC

Large Hadron Collider LHC
How do we study bulk QCD matter in the lab?

- We must heat and compress a large volume of QCD matter
- Done in the lab by colliding heavy nuclei at very high energies

Normal matter:

Large, hot, dense system: (Heavy Ion collisions)

No free quarks are seen, confined within hadron:
\[
\Delta v_0 \sim 1 \text{ fm}^3, \quad \rho_0 \sim 0.16 \text{ fm}^{-3}, \quad \varepsilon_0 \sim 0.15 \text{ GeV/fm}^3
\]

QGP
free quarks and gluons
in large volume

\[
\Delta v \sim 1000 \text{ fm}^3 = 1000 v_0
\]
\[
\rho \gg 3 \text{ fm}^{-3} \sim 20 \rho_0
\]
\[
\varepsilon \gg 3 \text{ GeV/fm}^3 \sim 20 \varepsilon_0
\]
Center of Mass Energy: Fixed target vs Collider

\[
E^2 - \vec{p}^2 c^2 = \left( m_0 c^2 \right)^2 \equiv s \quad \text{Lorentz invariant}
\]

Lorentz invariant can be used for a single particle and for the system of particles

\[
E_{\text{total}}^2 - \vec{p}_{\text{total}}^2 c^2 = \left( M_0 c^2 \right)^2 \equiv s
\]

… or using 4-vector notation (relativity):

\[
P = \begin{pmatrix} P^0 \\ P^1 \\ P^2 \\ P^3 \end{pmatrix} = \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}
\]

\[
\eta^{\mu \nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

\[
-|p|^2 = -p^\mu p_\mu = -\eta_{\mu \nu} p^\mu p^\nu = \left( E/c \right)^2 - (\vec{p})^2 = \left( m_0 c \right)^2
\]

**Exercise:** Calculate \(\sqrt{s}\) in a) and b) assuming \(m_1 = m_2\)
Typically at high energies: $E_1 \gg m_1, m_2$:

- e.g. $m_p = 0.938$ GeV/c

**Center of Mass Frame**

- Before Collision
  - $\vec{p}_1 + \vec{p}_2 = 0$

**Laboratory Frame**

- Before Collision
  - $\vec{p}_1 \rightarrow \vec{p}_1$
  - $\vec{p}_2 \rightarrow \vec{p}_2 = 0$

**Center of Mass Energy: Fixed target vs Collider**

$E_{total}^2 - \vec{p}_{total}^2 c^2 = (M_0 c^2)^2 \equiv s$

$= (E_1^{CM} + E_2^{CM})^2 - 0 = (E_{total}^{CM})^2$

if $m_1 = m_2 \Rightarrow E_1^{CM} = E_2^{CM}$

$\sqrt{s} = E_{total}^{CM} = 2E_1^{CM}$

$E_{total}^2 - \vec{p}_{total}^2 c^2 = (M_0 c^2)^2 \equiv s$

$= (E_1^{lab} + E_2^{lab})^2 - (c p_1^{lab})^2$

$= 2E_1^{lab} m_0^2 c^2 + (m_0^1 c^2)^2 + (m_0^2 c^2)^2$

if $E_1^{lab} >> m_0^{1,2} c^2 \Rightarrow \sqrt{s} \cong \sqrt{2E_1^{lab} m_0^2 c^2}$
Nuclear Force (Model) vs Strong Interactions (fundamental)

- Acts at distances of ~1fm
- At ~ 2.5 fm – decreases
- At ~ 0.7fm - repulsive
  (physical sizes of nuclei)

\[ \pi^0 : \frac{1}{\sqrt{2}} (d\bar{d} - u\bar{u}) \]
\[ \pi^+ : u\bar{d} \]
\[ \pi^- : \bar{u}d \]

\(\pi\)'s – part of “meson Octet”
  (hadron classification)

Green, red and blue – quark’s (3) colors
Nuclear Physics (N.P.)

N.P. is used to study fundamental laws of physics. Forces responsible for nuclear properties are: strong, e-m, and week.

However:

N.P. lacks a coherent theoretical formulation that would permit us to analyze and interpret all phenomena in a fundamental way (e.g. atomic physics and QED). N.P. needs to be discussed in a phenomenological way, using a different formulation to describe each different type of phenomenon, such as a decay, b decay, direct reactions or fission. Some of the most fundamental problems of N.P. such as the exact nature of the forces that hold the nucleus together, are yet unresolved (although much progress has been made)

S.M. Wong, “Introduction to Nuclear Physics”
Nucleon Structure

Small distances ....

How? Scattering experiment (Rutherford, again …)
Use high energy beam of particles to look inside the nucleon
Electron-Proton (or nucleus) Scattering

Elastic scattering

Inelastic scattering

$Q^2 \sim \text{mass of the virtual photon}$

Important variable

$1\text{fm}=10^{-15}\text{ m}\sim\text{size of a nucleon}$
Elastic Scattering

\[ x^\mu = (t, \vec{x}) \quad \text{4-vectors} \]
\[ p^\mu = (E, \vec{p}) \]

**Convention: c=1**

Scalar product of two 4-vectors:

\[ x \cdot p = \]
\[ p^2 = \]
\[ q^2 = (p - p')^2 = \]
\[ E \sim |\vec{p}| \quad \text{if} \ E >> m \]
\[ -Q^2 \equiv q^2 = \]
Elastic Scattering

\[ x^\mu = (t, \vec{x}) \quad \text{4-vectors} \]
\[ p^\mu = (E, \vec{p}) \]

Scalar product of two 4-vectors:

\[ x \cdot p = tE - \vec{x} \cdot \vec{p} \]
\[ p^2 = p \cdot p = E^2 - \vec{p}^2 = m^2 \]
\[ q^2 = (p - p')^2 = p^2 + p'^2 - 2 p \cdot p' = 2m^2 - 2(EE' - \vec{p} \vec{p}') \]

\[ E \sim |\vec{p}| \quad \text{if} \quad E \gg m \]
\[ -Q^2 \equiv q^2 = 2m^2 - 2EE'(1 - \cos \theta) \]

Convention: \( c=1 \)
Elastic Scattering

\[ x'' = (t, \vec{x}) \]
\[ p'' = (E, \vec{p}) \]

**4-vectors**

**Scalar product of two 4-vectors:**

\[ x \cdot p = tE - \vec{x} \cdot \vec{p} \]
\[ p^2 = p \cdot p = E^2 - \vec{p}^2 = m^2 \]
\[ q^2 = (p - p')^2 = p^2 + p'^2 - 2p \cdot p' = 2m^2 - 2(EE' - \vec{p}\vec{p}') \]

\[ E \sim |\vec{p}| \quad \text{if} \quad E \gg m \]
\[ -Q^2 = q^2 = 2m^2 - 2EE'(1 - \cos \theta) \]

**Convention: c=1**

\[ p = (E, \vec{p}) \]
\[ P = (M, \vec{0}) \]

\[ p + P = p' + P' \]

\[ E' = \frac{E}{1 + E/M (1 - \cos \theta)} \]

E - E' = recoil energy of the target

If the target is point-like then elastic scattering is the only possibility.
Elastic scattering experiment

\[ e^- + p \rightarrow e^- + p \quad E_{\text{beam}} = 529 \text{ MeV} \quad \theta = 75^\circ \]

\[ E' = \frac{E}{1 + E/M(1 - \cos \theta)} \]
\[ Q^2 \sim 2EE'(1 - \cos \theta) \]

\[ E' = \frac{529 \text{ MeV}}{1 + 529/938(1 - \cos 75^\circ)} = 373 \text{ MeV} \]
\[ Q^2 = 2 \times 529 \text{ MeV} \times 373 \text{ MeV} \times (1 - \cos 75^\circ) = 0.3 \text{ GeV}^2 \]
The Rutherford Cross Section

Electron scattering off an atomic nucleus (heavy) with charge $Z e$

Recoil neglected: $E = E'$ and $|\vec{p}| = |\vec{p}'|$
Spin effects neglected, point like probe (electron) and point like target

$$\frac{d\sigma}{d\Omega}_{\text{Rutherford}} = \frac{(Ze^2)^2}{4E^2 \sin^4 \frac{\theta}{2}} \left[ = \frac{(Ze^2)^2}{(4\pi\varepsilon_0)4E^2 \sin^4 \frac{\theta}{2}} \right]$$

Derivation:
see mechanics (non-relativistic) text books

$$e^2 = \alpha (4\pi\varepsilon_0 \hbar c), \quad \alpha \equiv \alpha_{\text{em}}$$

$$\frac{d\sigma}{d\Omega}_{\text{Rutherford}} = \frac{Z^2 \alpha^2 E'^2}{|q|^4} \left[ = \frac{Z^2 \alpha^2 E'^2 (\hbar c)^2}{|q|^4} \right]$$

$$|\vec{q}| = 2|\vec{p}| \sin \frac{\theta}{2}$$
The Rutherford Cross Section

Electron scattering off an atomic nucleus (heavy) with charge $Z e$

Recoil neglected: $E = E'$ and $|\vec{p}| = |\vec{p}'|$

Spin effects neglected, point like probe (electron) and point like target

\[
\left( \frac{d\sigma}{d\Omega} \right)_{\text{Rutherford}} = \frac{(Ze^2)^2}{4E^2 \sin^4 \frac{\theta}{2}} \left[ = \frac{(Ze^2)^2}{(4\pi\varepsilon_0) 4E^2 \sin^4 \frac{\theta}{2}} \right]
\]

Derivation:
see mechanics (non-relativistic) text books

\[
\left( \frac{d\sigma}{d\Omega} \right)_{\text{Rutherford}} \sim |\text{Amplitude}|^2 \sim \left( e \cdot \frac{1}{|\vec{q}|} \cdot Ze \right)^2
\]

Feynman diagrams: exchange of a virtual photon (factor $|q|^{-2}$ in the amplitude) coupling to the charged particles with strength proportional to their charges.
The Mott Cross Section

Electron scattering off an atomic nucleus (heavy) with charge Ze

Recoil neglected: $E = E'$ and $|\vec{p}| = |\vec{p'}|$  
Spin effects included (electron spin=1/2, point-charge charge spin=0 target)

$\left(\frac{d\sigma}{d\Omega}\right)^*_\text{Mott} = \left(\frac{d\sigma}{d\Omega}\right)_\text{Rutherford} \times \left(1 - \beta^2 \sin^2 \frac{\theta}{2}\right) \quad \beta = \nu/c$

drops rapidly at large scattering angles

$\beta \to 1 \quad \left(\frac{d\sigma}{d\Omega}\right)^*_\text{Mott} = \left(\frac{d\sigma}{d\Omega}\right)_\text{Rutherford} \cdot \cos^2 \theta \sqrt{\frac{4Z^2\alpha^2E'^2}{|q|^4}} \cdot \cos^2 \frac{\theta}{2}$

$\left(\frac{d\sigma}{d\Omega}\right)^*_\text{Mott} \sim \frac{\alpha^2E'^2}{|q|^4} \cos^2 \theta \quad \theta \to \pi \Rightarrow \left(\frac{d\sigma}{d\Omega}\right)^*_\text{Mott} \sim 0$

due to electron helicity conservation (spinless target)
Probing the nucleus

- Au and C – not point-like
- H – not point-like, either

Electron scattering from the proton at an incident energy of 188 MeV. Curve (a) shows the theoretical Mott curve for a spinless point proton. Curve (b) shows the theoretical curve for a point proton with a Dirac magnetic moment alone. Curve (c) shows the theoretical behavior of a point proton having the anomalous Pauli contribution in addition to the Dirac value of the magnetic moment. The deviation of the experimental curve from the Curve (c) represents the effect of form factors for the proton and indicates structure within the proton. The best fit in this figure indicates an rms radius close to 0.7 \times 10^{-13} \text{ cm}.

The Mott Cross Section

Electron scattering off an atomic nucleus (heavy) with charge $Ze$
Spin effects included (electron spin=1/2, point-charge charge spin=0 target)

- Recoil neglected: $E = E'$ and $|\vec{p}| = |\vec{p}'|$

$$\left(\frac{d\sigma}{d\Omega}\right)_{Mott}^* = \left(\frac{d\sigma}{d\Omega}\right)_{Rutherford} \cdot \cos^2 \frac{\theta}{2} \sim \left[ \frac{4Z^2\alpha^2E'^2}{|q|^4} \right] \cdot \cos^2 \frac{\theta}{2}$$

- Recoil not neglected:

$$\left(\frac{d\sigma}{d\Omega}\right)_{Mott} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott}^* \cdot \frac{E'}{E}$$

$$q^2 = (p - p')^2 \approx -4EE'\sin^2 \frac{\theta}{2}$$

$$Q^2 = -q^2$$
Nuclear Form Factors

In elastic scattering experiments with nuclei or nucleons:

- target = point like
  \[ \frac{d\sigma}{d\Omega}_{\text{exp}} = \left( \frac{d\sigma}{d\Omega} \right)_{\text{Mott}}^* \]

- target = not point like
  \[ \frac{d\sigma}{d\Omega}_{\text{exp}} = \left( \frac{d\sigma}{d\Omega} \right)_{\text{Mott}}^* \cdot |F(\vec{q}^2)|^2 \]

Charge distribution (nuclei or nucleon) \( \rho(\vec{x}) = Ze \cdot f(\vec{x}) \quad \int f(\vec{x})d^3x = 1 \)
and its Fourier transform
\[ F(\vec{q}^2) = \int e^{i\vec{q} \cdot \vec{x}/\hbar} f(\vec{x})d^3x \]
Cross Section Measurements

\[
\left( \frac{d\sigma}{d\Omega} \right)_{\text{exp}} = \left( \frac{d\sigma}{d\Omega} \right)_{\text{Mott}}^* \cdot \left| F(\vec{q}^2) \right|^2
\]

Superposition of a fast fall-off with a typical pattern associated with the form factor.
**Nuclear Form Factors**

Charge distribution (nuclei or nucleon) \( \rho(\bar{x}) = Ze \cdot f(\bar{x}) \)

and its Fourier transform \( \mathcal{F}(\bar{q}^2) = \int e^{i\bar{q} \cdot \bar{x}/\hbar} f(\bar{x}) d^3x \)

For a spherically symmetric cases: \( f(\bar{x}) = f(r) \)

\[
\mathcal{F}(\bar{q}^2) = 4\pi \int f(r) \frac{\sin(qr/\hbar)}{qr/\hbar} r^2 dr
\]

\[
1 = \int f(\bar{x})d^3x = 4\pi \int f(r)r^2 dr \quad \text{f= probability density}
\]

\[
f(r) = \frac{1}{(2\pi)^3} \int e^{-i\bar{q} \cdot \bar{x}/\hbar} \mathcal{F}(\bar{q}^2) d^3q
\]

**Example:** Calculate form factor for \( \rho(r) = \text{const} \quad r \leq a \)

\[
= 0 \quad r > a
\]
Nuclear Form Factors

Charge distribution (nuclei or nucleon) \( \rho(\vec{x}) = Ze \cdot f(\vec{x}) \)
and its Fourier transform \( F(\vec{q}^2) = \int e^{i\vec{q} \cdot \vec{x}/\hbar} f(\vec{x}) \, d^3x \)

For a spherically symmetric cases: \( f(\vec{x}) = f(r) \)

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\[
f(r) = \frac{1}{(2\pi)^3} \int e^{-i\vec{q} \cdot \vec{x}/\hbar} F(\vec{q}^2) \, d^3q
\]

Example: Calculate form factor for \( \rho(r) = \text{const} \)

\[
f(r) = \text{const} = \frac{\rho}{Ze} = \frac{V}{4/3\pi a^3} = \frac{1}{[4/3\pi a^3]}
\]

\( r \leq a \)

\( = 0 \quad r > a \)
See you on Wednesday 02/18!

Questions, comments:
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Or

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