1. A deuteron is bound state of proton and neutron \(m_p \sim m_n\). It can be modeled as one particle of reduced mass \((m = m_p m_n/(m_p + m_n))\) in a central square well potential of width \(b\) and depth \(-V_0\), as shown in the figure. For the deuteron there is only one bound state, which has no orbital angular momentum.

\[ V(r) \]

\[ b \]

\[ V_0 \]

\[ E_B \]

\[ r \]

\[ a) \quad (20\text{ points}) \text{ Write down the Schroedinger equation for the radial wave function } u(r) = r R(r) \text{ for the two regions } r < b \text{ and } r > b. \text{ Give the solution for } u(r) \text{ in each region. Reduce the number of coefficients as, but do not attempt to calculate them.} \]

\[ b) \quad (10) \text{ Show that from the general solution you get a relation between } E_B, V_0, b \text{ of the form:} \]

\[ \tan(kb) = \frac{k}{k'} \]

Where \(k, k'\) are functions of \(m, E_B, V_0\). Write down the expression in terms of \(E_B, V_0, b\).

2. Show that in a central potential orbital angular momentum and total energy of a state are compatible variables. (It is sufficient to show this for one component, say \(L_z\), of the angular momentum operator. Make use of the fact that \(r_z\) commutes with \(r_x, r_y, r_z, \partial/\partial x, \partial/\partial y\) but not with \(\partial/\partial z\) and that \(\partial/\partial z\) commutes with \(r_x, r_y, \partial/\partial x, \partial/\partial y, \partial/\partial z\) but not with \(z\). Or expressed in terms of commutators: \([r_x, r_y] = [p_x, p_y] = 0\) while \([r_x, p_y] = i\hbar \delta_{xy}\).)
Sky 204E 2nd midterm exam

\[ a) \quad -\frac{k^2}{2m} \frac{d^2u}{dr^2} + \left( V(r) + \frac{k^2}{2m} \frac{\ell(\ell + 1)}{r^2} \right) u = E_s u \]

with \( u(r) = r R(r) \)

for diatomic ground state \( L = 0 \)

\[ \Rightarrow \quad \frac{d^2u}{dr^2} = \frac{2m}{k^2} (V(r) - E_0) u \]

\[ \text{Region I: } r < b \quad V(r) = -V_0 \]

\[ \frac{d^2u}{dr^2} = -\frac{2m}{k^2} (V_0 + E_0) u \]

\[ k^2 = \frac{2m}{k^2} (V_0 + E_0) \quad \text{note } V_0 > |E_0| \]

\[ \Rightarrow \quad \frac{d^2u}{dr^2} = -k^2 u \]

\[ \text{Solution: } \quad U_a(r) = A \sin(kr) + B \cos(kr) \]

Since \( u = r R \cos(kr) \) changes for \( r \to 0 \)

\( U(r) \) normalizable \( \Rightarrow \) \( B = 0 \)

\[ \Rightarrow \quad U_a(r) = A \sin(kr) \]
Region II: \( r > b \) \( V(r) = 0 \)

\[
\frac{d^2 u}{dr^2} = - \frac{2m_f e_{\|}}{k_r^2} u
\]

Since \( \frac{k_r^2}{k_f^2} < 0 \), \( k_r^2 = - \frac{2m_f e_{\|}}{k_f^2} \) is real.

\[
\frac{d^2 u}{dr^2} = k_r^2 u
\]

Solution: \( u(\|) = C e^{-k_r r} + D e^{k_r r} \)

for \( r \to \infty \) \( e^{k_r r} \) diverges \( \Rightarrow D = 0 \)

\[
\boxed{u(\|) = Ce^{-k_r r}}
\]

b) Use boundary conditions to get relation between \( a \) and \( b \):

\( F = \frac{1}{b} - \frac{1}{a} \)

1. \( u(\|) = u(\|) \)

\[
A C \sin(k b) = C e^{-k b}
\]

2. \( u'(\|) = u'(\|) \)

\[
A k \cos(k b) = -C k' e^{-k b}
\]

Combining (1) & (2) gives \( \frac{1}{k} \tan(k b) = -\frac{1}{k'} \)

or \( \tan(k b) = -\frac{k}{k'} \)
inserting \( k = \sqrt{\frac{2m(V_0 - E)}{k^2}} \) and \( k' = \sqrt{\frac{-2mE_0}{k^2}} \)

\[
\tan \sqrt{\frac{2m(V_0 + E)k^2}{h^2}} = -\frac{\sqrt{V_0 + E_0}}{-E_0}
\]
Show that the total energy and angular momentum are compatible variables in a central potential.

To be compatible variables, equations must commute

\[ [H, L] = 0 \]

It is sufficient to show this for one coordinate of \( L \).

Since \( H \) is symmetric in \( x, y, z \) in central potential,

\[ L_z = \frac{\partial}{\partial y} x \frac{\partial}{\partial z} - x \frac{\partial}{\partial y} \]

\[ H = -\frac{\hbar^2}{2m} \nabla^2 + V(r) = -\frac{\hbar^2}{2m} + V(r) \]

Since \( p^2 = \sum p_i^2 = p_x^2 + p_y^2 + p_z^2 \), we need to show

\[ [p_i, L_z] + [p_j, L_z] + [p_k, L_z] = [V, L_z] = 0 \]

\[ [p^2, L_z] = [p_x, x p_y - y p_x] = [p_x^2, x] - [p_x^2, y] = 0 \]

\[ = 2 \{ p_x, x \} p_x \]

\[ = 2 \hbar \rho_x \rho_y \]
\[
\left[ \rho_x^2, x\rho_y - y\rho_x \right] = \left[ \rho_y, x\rho_y \right] - \left[ \rho_y^2, y\rho_x \right]
\]
\[
= - \left[ \rho_y, x\rho_y \right] - y\left[ \rho_y^2, y\rho_x \right]
\]
\[
= - 2\left[ \rho_y \right] \rho_x = -2\lambda \rho_x \rho_y
\]
\[
\left[ \rho_x^2, x\rho_y - y\rho_x \right] = 0
\]

So, \( \left[ \rho^2 L_2 \right] = 0 \) and we are left to show

\[
\left[ V, x\rho_y - y\rho_x \right] \psi = \frac{\hbar}{i} \left[ V, \frac{\partial}{\partial y} \right] \psi - \frac{\hbar}{i} \left[ V, \frac{\partial}{\partial x} \right] \psi
\]

\[
= \frac{\hbar}{i} V_{\frac{\partial^2}{\partial y^2}} - \frac{\hbar}{i} x \frac{\partial}{\partial y} \left( \frac{\partial}{\partial y} \psi \right) - \frac{\hbar}{i} \frac{\partial}{\partial x} \frac{\partial}{\partial y} - \frac{\hbar}{i} \frac{\partial}{\partial y} \frac{\partial}{\partial x} \psi
\]

\[
= - \frac{\hbar}{i} x \frac{\partial^2 \psi}{\partial y^2} + \frac{\hbar}{i} \frac{\partial^2 \psi}{\partial x \partial y}
\]

Now, \( \frac{\partial V}{\partial y} = \frac{\partial \sigma}{\partial y} \frac{\partial V}{\partial \sigma} = 2Y \frac{\partial V}{\partial \sigma} \)

And \( \frac{\partial V}{\partial x} = \frac{\partial \sigma}{\partial x} \frac{\partial V}{\partial \sigma} = 2X \frac{\partial V}{\partial \sigma} \)

\[ \Rightarrow \left[ V, L_2 \right] = 0 \quad \text{q.e.d.} \]