Gravitational Waves

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03/02/2015
Outline

Introduction

Formulation

Sources of Gravitational Waves

Detection Methods

Conclusion
Why General Relativity? (GR)

- Explains precession of orbit of Mercury
- Predicts bending of light by the sun
- Generalization of special relativity to gravity
Introduction to GR

- 3 dim space → 4 dim spacetime
- Spacetime as Pseudo-Riemannian manifold
- Link between energy and curvature given by Einstein equations

Source: https://orbismediologicus.files.wordpress.com/2009/
Introduction to GR

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Metric

- Bi-linear non-degenerate product over the manifold.
- \( g(e_\mu, e_\nu) = g_{\mu\nu} \)
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- $g(e_\mu, e_\nu) = g_{\mu\nu}$

$$\eta_{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}$$

Source: http://themaclellans.com/timetravel.html
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Weak Field Approximation

For weak gravity, we have \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \) where \( h \) is first order.

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Coordinate Transformations

- Tensors transform in a specific way
- Lorentz transformations (LT) is Special Relativity (SR)
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\Lambda(\beta) = \begin{pmatrix}
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-\beta \gamma & \gamma & 0 & 0 \\
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- \(\eta_{\mu\nu}\) is invariant under boosts.
- \(h_{\bar{\alpha}\bar{\beta}} = \Lambda^\mu_{\bar{\alpha}} \Lambda^\nu_{\bar{\beta}} h_{\mu\nu}\)
- Under LT, \(h_{\mu\nu}\) transforms like a tensor field in flat space. (First order)
Gauge Transformation

- GR analogue of degree of freedom in metric (curvature) that leaves equations of motion (EoM) invariant.
- $x^{\bar{\alpha}} = x^{\alpha} + \xi^{\alpha}$ as a coordinate transformation
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Weak Field Approximation

Up to first order, \( h_{\mu\nu} \rightarrow h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} \)
Connection

- Derivative operator on tensor fields in a manifold is given by the connection,

- Define $\nabla_\mu \eta^\nu := \eta^\nu,\mu + \Gamma^\nu_{\mu\alpha} \eta^\alpha$ with the notation;

- $T^{\alpha\ldots}_{\beta\ldots,\mu} := \partial_\mu T^{\alpha\ldots}_{\beta\ldots}$ and $T^{\alpha\ldots}_{\beta\ldots;\mu} := \nabla_\mu T^{\alpha\ldots}_{\beta\ldots}$

- Condition for compatibility $(g_{\mu\nu;\alpha} = 0)$ defines a unique connection.
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- Condition for compatibility \((g_{\mu \nu; \alpha} = 0)\) defines a unique connection.

- \( \Gamma^\mu_{\nu \rho} = \frac{1}{2} g^{\mu \lambda} \left( g_{\nu \lambda, \rho} + g_{\rho \lambda, \nu} - g_{\nu \rho, \lambda} \right) \) is the Riemann connection.
Curvature

Source: http://jefferywinkler.com/tensors.html
Curvature

- Measure of how much vector fields deviate on straight lines.
- Riemann tensor; $R^{\mu}_{\nu\rho\sigma} = \Gamma^{\mu}_{\nu\rho,\sigma} - \Gamma^{\mu}_{\nu\sigma,\rho} + \Gamma^{\mu}_{\sigma\alpha} \Gamma^{\alpha}_{\nu\rho} - \Gamma^{\mu}_{\rho\alpha} \Gamma^{\alpha}_{\nu\sigma}$. 

Curvature tensor, to first order, is $R^{\alpha\beta}_{\mu\nu} = \frac{1}{2} (h^{\alpha\nu;\beta,\mu} + h^{\beta\mu;\alpha,\nu} - h^{\alpha\mu;\beta,\nu} - h^{\beta\nu;\alpha,\mu})$ and is invariant under Gauge transformations.
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- \( R_{[(\alpha\beta)(\mu\nu)]} = R_{\alpha\beta\mu\nu} \)
- \( R^\mu_{[\alpha\beta\gamma]} = 0 \)
- \( R^\mu_{\nu[\alpha\beta;\gamma]} = 0 \)

- Riemann tensor has \( \frac{n^2(n-1)^2}{12} \) indep. terms.
- Ricci tensor; \( R^\mu_{\nu} := R^\sigma_{\mu\sigma\nu} \) \( n(n-1) \) terms
- Ricci scalar (curvature); \( R := R^\alpha_{\alpha} \)
Curvature

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- Ricci scalar (curvature); \( R := R^\alpha_\alpha \)

Small Field Approximation

Curvature tensor, to first order, is
\[
R_{\alpha\beta\mu\nu} = \frac{1}{2} \left( h_{\alpha\nu,\beta,\mu} + h_{\beta\mu,\alpha,\nu} - h_{\alpha\mu,\beta,\nu} - h_{\beta\nu,\alpha,\mu} \right)
\]
and is invariant under Gauge transformations
Einstein Equations

- $G^{\mu\nu} := R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi T^{\mu\nu}$ is the Einstein equations.
- Useful quantity, Trace reverse; $\bar{h}^{\alpha\beta} := h^{\alpha\beta} - \frac{1}{2} \eta^{\alpha\beta} h$
- Lorentz gauge, $\bar{h}^{\mu\nu},_{\nu} = 0$
Einstein Equations

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- Lorentz gauge, \( \bar{h}^{\mu\nu,\nu} = 0 \)

Small Field Approximation

Lorentz gauge yields \( \Box \bar{h}^{\mu\nu} = -16\pi T^{\mu\nu} \) with an extra gauge freedom of \( \Box \eta^\mu = 0 \)
Wave solution

- Try solution $\tilde{h}^{\mu \nu} = A^{\mu \nu} \exp (i k_\alpha x^\alpha)$
- Condition of $k^\nu k_\nu = 0$ implies waves travel at the speed of light. (lightlike)
- Condition $A^{\alpha \beta} k_\beta = 0$ implies plane wave.
- Gauge freedom gives the tranverse traceless gauge; $A_{\mu}^{\mu} = 0$ & $A_{\alpha \beta} u^\alpha = 0$ for some 4-velocity.
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Small Field Approximation

\[
A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & A_\times & 0 \\ 0 & A_\times & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]
Geodesic Equation

- Paths of particles given by the geodesic equation.
  \[ \frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\sigma\rho} \frac{dx^\sigma}{d\lambda} \frac{dx^\rho}{d\lambda} = 0 \]
  - Initially, acceleration is zero and no \( \Gamma_0 \) terms, thus coordinates don’t change.
  - However, metric changes, thus proper distance changes.
Stretching of Space

- Particles separated by $\epsilon$ (in x-dir) obey the equation

$$\frac{\partial^2}{\partial t^2} \eta^i = \frac{1}{2} \epsilon \frac{\partial^2}{\partial t^2} h_{xi}$$

- Two polarizations ('×' and '+' polarization) that are 45°

- Contrast to EM waves with 90° and vector potential $A^\nu$ instead of a tensor potential.

Visualization of elliptical gravitational waves

(a) and (b) show the elliptical nature of gravitational waves, while (c) represents a circular wavefront.

(d) and (e) illustrate the movement of wavefronts in the z-direction, and (f) shows the interaction of wavefronts in different planes.

Source:
http://relativity.livingreviews.org/Articles/lrr-2013-7/GWPolStates.png
http://phys23p.sl.psu.edu/phys_anim/astro/indexer...
Unlikely sources

- Man-made sources
  - detection requires distance from source
  - energy radiated is miniscule
Want to consider source, $T^{\mu \nu} \neq 0$

Define a quadrupole moment $I^{lm} := 2 \int T^{lm} \, d^3 \mathbf{x}$ and

$I^{lm} := I^{lm} - \frac{1}{3} \delta^{lm} I^k_k$

The relation $\square \bar{h}_{\mu \nu} = -16 \pi T_{\mu \nu}$, if assuming a small region of sinusoidal dependence ($T_{\mu \nu} = S_{\mu \nu}(x^i) \exp^{-i\Omega t}$) gives;

\[
\begin{align*}
h_{xx} &= -h_{yy} = -\Omega^2 \left( I_{xx} - I_{yy} \right) \exp^{i\Omega r} / r \\
h_{xy} &= -\Omega^2 I_{xy} \exp^{i\Omega r} / r
\end{align*}
\]
Gravitational Collapse

- Waves originating from (asymmetrical) collapse of core collapse supernovae
- Only information of the core from neutrino and GW
- Still poorly understood
- Modelling difficult
- Waveform predictions not possible
- Not very luminous (OK if within our galaxy)
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Conclusion

Not a likely candidate for detection
Binary Systems

- Orbiting systems with energy loss.
- Indirect measurement method

Binary systems

- 4 detected NS-NS (neutron star) pairs in our galaxy
- Increasing merger $\Omega$ results in increase in signal frequency (chirp)
- BH-BH (Black hole) mergers have unique waveforms (simulations)
- Chirping allows distance measurements (standard siren)
- Mostly circular orbit, as eccentricity is radiated away.

Source: Hong J., Lee H. (2014)
Cosmological Sources

- Many theoretical predictions with varying intensities
- Will most probably remain below the noise
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Conclusion

Most likely not a good source, and will not be observed. (directly)
Why do we want to detect GW?

- Direct info from supernovae (SN)
- May probe pre-recombination era
- Allows further testing of GR.
- May allow dark matter probing
Detection methods

- Resonant mass detectors (bar detectors)
  - Laser interferometer
  - Beam detectors
Bar Detectors

- Resonant mass detectors.
- For a spring, a GW induces the EoM; (classical driven damped HO)
  \[ \partial_t^2 \eta + 2\gamma \partial_t \eta + \omega_0^2 \eta = \frac{1}{2} l_0 \partial_t^2 h_{xx} \]
- Large bars with elasticity as oscillators
Bar Detectors

- Resonant mass detectors.
- For a spring, a GW induces the EoM; (classical driven damped HO)
  \[ \frac{d^2}{dt^2} \eta + 2\gamma \frac{d}{dt} \eta + \omega_0^2 \eta = \frac{1}{2} l_0 \frac{\partial^2}{\partial t^2} h_{xx} \]
- Large bars with elasticity as oscillators
- Likely sources will excite amplitudes \( \sim 10^{-15} m \).
- Thermal and mechanical noise
- Ultra-low-noise low-temperature superconducting detectors (SQUIDS)
Laser Interferometry

- Ground based
- Photon traveling along a GW has an effective coordinate speed lowered below 1.
- Under GW, the frequency is redshifted

\[
\frac{d t_{\text{return}}}{d t_{\text{start}}} = \frac{\nu_{\text{return}}}{\nu_{\text{start}}} = 1 + \frac{1}{2} \left[ h_+ (t_{\text{start}} + 2L) - h_+ (t_{\text{start}}) \right]
\]

- Two interferometers detectors are there, LIGO (US) and VIRGO (Italy).
- Run from 40 Hz to 1 kHz.

Source: www.ligo.caltech.edu
Beam Detectors

- Same principle as interferometers, but larger distances.
- Proposed as spacecraft.
- Currently, with $10^{-19}$ precision on time, can’t measure amplitudes $< 10^{-20}$ which is the expected range.
- LISA, 3 spacecraft at 1 AU as an equilateral triangle
  → Frequency range mHz.
  → Drag-free operation
  → Multiple satellites measures polarization.
Conclusion

- GR generally leads to GW, not predicted by Newtonian gravity.
- Binary systems have been detected to lose energy with magnitudes estimated correctly by GR.
- Detectors are in place and operational, along with better detectors being planned.
- Most likely source of GW are binary pulsars and BHs.
Sources

1. Schutz, A First Course in General Relativity