Imaginary Numbers

- def. $i^2 = -1$
  
  $i = \sqrt{-1}$
  
  $i^6 = ?$

- a complex number: $z = x + iy$
  
  $x$ – real part of $z$, $x = \text{Re}(z)$
  
  $y$ – imaginary part of $z$, $y = \text{Im}(z)$
  
  real number $s=x$, e.g. $s=2$
  
  imaginary number $q=iy$ e.g. $q=3i$

- a complex conjugate: $z^* = (x + iy)^* = x - iy$
  
  i.e. replace $i \rightarrow -i$

- $|z|^2 = zz^* = (x + iy)(x - iy) = x^2 + y^2$
Math (reminder)

\[ i^2 = -1 \]

Imaginary Numbers

\[ z_1 = (x_1 + iy_1) \]
\[ z_2 = (x_2 + iy_2) \]

- **Addition (and subtraction)**

\[ z_1 \pm z_2 = (x_1 + iy_1) \pm (x_2 + iy_2) = (x_1 \pm x_2) \pm i(y_1 \pm y_2) \]

- **Multiplication (similarly division)**

\[ z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1x_2 - y_1y_2) + (x_2y_1 + x_1y_2)i \]

- **Absolute value:**

\[ z = x + iy \quad z^* = (x + iy)^* = x - iy \]

\[ |z| = \sqrt{|z|^2} = \sqrt{z^*z} = \sqrt{x^2 + y^2} \geq 0 \]

\[ |z|^2 = x^2 + y^2 \geq 0 \]

\[ |z|^2 = zz^* = (x + iy)(x - iy) = x^2 + y^2 \]

Always a real number!
Imaginary Numbers

$z = x + iy$ can be represented as a point on a complex plane.

**Absolute value:**
\[ r = |z| = \sqrt{|z|^2} = \sqrt{z^*z} = \sqrt{x^2 + y^2} \geq 0 \]

Always a real number!

**Note:**
\[ |z|^2 \neq z^2 \]
Example 1  Perform the indicated operation and write the answers in standard form.
   (a) \((-4 + 7i) + (5 - 10i)\)
   (b) \((4 + 12i) - (3 - 15i)\)
   (c) \(5i - (-9 + i)\)

Example 2  Multiply each of the following and write the answers in standard form.
   (a) \(7i(-5 + 2i)\)
   (b) \((1 - 5i)(-9 + 2i)\)
   (c) \((4 + i)(2 + 3i)\)
   (d) \((1 - 8i)(1 + 8i)\)

Example 3  Write each of the following in standard form.
   (a) \(\frac{3 - i}{2 + 7i}\)
   (b) \(\frac{3}{9 - i}\)
   (c) \(\frac{-8i}{1 + 2i}\)
   (d) \(\frac{6 - 9i}{2i}\)

Example 4  Multiply the following and write the answer in standard form.
   \[ (2 - \sqrt{-100})(1 + \sqrt{-3}) \]
$z = x + iy$ can be represented as a point on a complex plane

$x = r \cos \theta, \ y = r \sin \theta$

$r = \sqrt{x^2 + y^2}, \ \tan \theta = \frac{y}{x}$

$z = r(\cos \theta + i \sin \theta)$

$|z| = \sqrt{zz^*} = \sqrt{x^2 + y^2} = r$
Complex Exponential

\[ z = x + iy \]

\[ z = r(\cos \theta + i \sin \theta) = re^{i\theta} \]

Note: \[
\cos \theta = \text{Re}(e^{i\theta}) \quad \cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \\
\sin \theta = \text{Im}(e^{i\theta}) \quad \sin \theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})
\]

Exercise 5:

a) Show \[ e^{i\theta} = \cos \theta + i \sin \theta \]

Exercise 6:

Show \[ \cos(A + B) = \cos A \cos B - \sin A \sin B \] using the complex exponential.