The Special Theory of Relativity

- Time dilation
- Minkowski (spacetime) diagrams
- Lorentz invariant: spacetime interval
- Causality
- Relativistic Dynamics
Time Dilation

A clock in a moving frame $S'$

The time to take a light to go from a laser to the mirror and back:

$$\Delta t' = \frac{2d}{c}$$
The time to take a light to go from a laser to the mirror and back:

\[
\left(\frac{c\Delta t}{2}\right)^2 = \left(\frac{v\Delta t}{2}\right)^2 + d^2
\]

\[
\Delta t = \frac{2d}{\sqrt{c^2 - v^2}} = \frac{1}{\sqrt{1 - v^2/c^2}} \times \frac{2d}{c} = \gamma \Delta t'
\]

\[\Delta t = \gamma \Delta t'\]
Time Dilation

The time interval $\Delta t$ measured by the observer moving with respect to the clock is longer than the time interval $\Delta t'$ measured by the observer at rest with respect to the clock.

$$\Delta t = \gamma \Delta t'$$

“A moving clock runs slower than a clock at rest by a factor of $\gamma$.”

$\Delta t'$ – proper time. It is always the time measured by an observer moving along with the clock

*Time dilation is a real phenomenon - it has been observed.*
Muons

Muon (µ) – a point-like elementary particle (lepton) (“heavier electron”)

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Mass of a muon $m_\mu \sim 105 \text{ MeV}/c^2$ (~200 times heavier than electron).

Muons are not stable (they can decay), unlike electrons. Muon’s mean lifetime $\sim 2.2 \ \mu s$.

An example of a decay channel:

$$\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu.$$  

electron

(anti-)neutrino
Muon’s lifetime
- in the muon rest frame and in the lab frame

\[
\nu = 0 \\
\tau_0 = 2.2 \, \mu s
\]

\[
\nu = 0.995c \\
\tau = 22 \, \mu s = 10\tau_0
\]

\[
\nu = 0.99995c \\
\tau = 220 \, \mu s = 100\tau_0
\]

The mean lifetime of a muon in its own reference frame, called the proper lifetime, is \( \tau_0 = 2.2 \, \mu s \). In a frame moving at velocity \( \nu \) with respect to that proper frame, the lifetime is \( \tau = \gamma \tau_0 \), where \( \gamma \) is the time dilation factor.

Mean lifetime \( \tau \) as measured in laboratory frame
Muon’s lifetime
- in the muon rest frame and in the lab frame

For muons travelling with speed 0.99c:

- An observer in the muons reference frame measures what is commonly called “the proper lifetime”, an Earth-based observer measures what is commonly referred to as “the proper height” of the mountain.
- In the muon’s reference frame there is no time dilation, but the distance of travel is observed shorter when measured in this frame.
- In the reference frame of the observer on Earth, there is time dilation but the distance of travel is measured to be the proper height of the mountain.
- The outcome of the experiment/process (a muon decay) doesn’t depend on the reference frame!
Practice problem:

Muons are produced when cosmic rays strike in the upper atmosphere. A muon travels with a speed of 0.99c the distance of 5km (in the Earth coordinate frame) from the moment it got produced to the moment it decayed.

a) What is the lifetime of this muon measured by an observer on Earth?

b) What is the lifetime of this muon measured in its rest frame?

c) What is the depth $\lambda$ of the atmosphere measured in the muon’s rest frame?
Minkowski Diagrams (also called spacetime diagrams)

The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a union of the two will preserve an independent reality. (Hermann Minkowski, 1908, in an address to the Assembly of German Natural Scientists and Physicians)
A Minkowski diagram showing the complete history or world line of a one-dimensional motion in a given frame.

A spacetime diagram showing the position of a particle in one dimension at consecutive times. The path showing the complete history of the particle is called the world line of the particle. An event E has coordinates \((x, t)\) in frame S and coordinates \((x', t')\) in \(S'\).
Minkowski diagrams can be used to classify the universe of spacetime and clarify whether or not one event could be the cause of another.

Classification of one-dimensional spacetime into past, future, and elsewhere regions. A particle with world line passing through O cannot reach regions marked elsewhere.
Lorentz Invariant

Most general case:

$$s^2 = (ct)^2 - x^2 - y^2 - z^2 = (ct')^2 - x'^2 - y'^2 - z'^2$$

*y=z=y'=z'=0* :

$$s^2 = (ct)^2 - x^2 = (ct')^2 - x'^2$$

$s^2$ invariant means it has the same value in S and S' frame (i.e. it does not depend on the frame)
Point Events and Their Transformations - Lorentz Invariant

Assumption:
S’ moves with a speed v with respect to S frame along x direction.

Two different point events \( P_1 \) and \( P_2 \), each describable in either S or S’

\[
\begin{align*}
  x'_1 &= \gamma(x_1 - vt_1) \\
  t'_1 &= \gamma(t_1 - vx_1/c^2)
\end{align*}
\]
\( P_1 \)

\[
\begin{align*}
  x'_2 &= \gamma(x_2 - vt_2) \\
  t'_2 &= \gamma(t_2 - vx_2/c^2)
\end{align*}
\]
\( P_2 \)

Spacetime interval between two events

\[
\Delta s^2 = s^2_{P_1 P_2} = (\Delta ct)^2 - (\Delta x)^2 = \left( c(t_2 - t_1) \right)^2 - (x_2 - x_1)^2
\]

\[
\Delta s'^2 = s'^2_{P_1 P_2} = (\Delta ct')^2 - (\Delta x')^2 = \left( c(t'_2 - t'_1) \right)^2 - (x'_2 - x'_1)^2
\]

\[
\Delta s'^2 = (\Delta ct)^2 - (\Delta x)^2 \implies \Delta s'^2 = \Delta s^2
\]

The “distance” in spacetime does not depend on the frame. It is INVARIANT under Lorentz transformations.
Spacetime interval $\Delta s^2$ between two events can be used to classify whether one event could be caused by another.

Consider 3 pairs of events in spacetime:
1) V and W
2) A and B
3) C and D

a) Could V cause W? Yes: $\Delta s^2 = (\Delta ct)^2 - (\Delta x)^2 > 0$ i.e. $\Delta ct > |\Delta x| \Rightarrow \frac{|\Delta x|}{\Delta t} < c$

b) Could A cause B? Yes: $\Delta s^2 = (\Delta ct)^2 - (\Delta x)^2 = 0$ i.e. $\Delta ct = |\Delta x| \Rightarrow \frac{|\Delta x|}{\Delta t} = c$

c) Could C cause D? No: $\Delta s^2 = (\Delta ct)^2 - (\Delta x)^2 < 0$ i.e. $\Delta ct < |\Delta x| \Rightarrow \frac{|\Delta x|}{\Delta t} > c$
Spacetime interval $\Delta s^2$ between two events can be used to classify whether one event could be caused by another

$$\Delta s^2_{P_1P_2} = (\Delta ct)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = (ct')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2$$

One can find a coordinate system where 2 events will take place

- at the same point in space if: $\Delta s^2_{R_{P_2}} > 0$  
  Events $P_1$ and $P_2$ can be connected (their order is always the same)  
  e.g. $P_1$ and $P_2$ - movement of a particle (proper time)

- at the same time if: $\Delta s^2_{R_{P_2}} < 0$  
  Events $P_1$ and $P_2$ can not be connected (their order depends on the frame)  
  e.g. length measurement of two ends of a stick

If $\Delta s^2_{P_1P_2} = 0$ then there is no such a coordinate system in which 2 events can happen at the same time or at the same place (events $P_1$ and $P_2$ can only be connected by a light signal)  
null interval
Relativistic Dynamics

relativistic mass, energy and momentum

conservation of relativistic momentum and energy
Reminder: in classical mechanics

\[ \sum_i \vec{p}_i = \sum_i m_i \vec{v}_i \]

Total momentum of an isolated system is conserved.

**Special relativity:**
Since the laws of physics are the same in all inertial frames, we expect the total momentum of an isolated system to be conserved in frame S as well as in frame S’ (which moves with a relativistic speed \(v\) with respect to the S frame). This implies that the definition of momentum is redefined in special relativity.*

*) not shown here
Relativistic Mass, Momentum and Energy

\[ m_0 \text{ particle’s “rest” mass} \]
\[ \vec{v} \text{ particle’s velocity} \]
\[ E \text{ particle’s energy} \]

Notation used in the textbook (in this lecture we will follow this notation)

Relativistic dynamics:

\[ \gamma = \left(1 - \frac{\vec{v}^2}{c^2}\right)^{-1/2} \]
\[ m(v) = \gamma m_0 \]
\[ \vec{p} = \gamma m_0 \vec{v} \]
\[ E = \gamma m_0 c^2 \]

Note:
\[ \gamma \] has the same functional form as the \[ \gamma \] in the Lorentz transformation, but here \[ \gamma \] contains the particle speed, while in the Lorentz transformation, \[ \gamma \] contains the relative speed of the two frames.
Relativistic Mass, Momentum and Energy

\[ E = \gamma m_0 c^2 \]
\[ E = E_{\text{kin}} + m_0 c^2 \]

Total energy is the sum of kinetic energy and “rest” energy (term which is independent of the speed)

\[ E_{\text{kin}} = \gamma m_0 c^2 - m_0 c^2 \]
\[ = m_0 c^2 \left[ \left(1 - \frac{v^2}{c^2} \right)^{-1/2} - 1 \right] \quad \rightarrow \quad m_0 c^2 \left( \frac{1}{2} \frac{v^2}{c^2} \right) = \frac{1}{2} m_0 v^2 \]

Classical expression for kinetic energy
A. Einstein’s mass–energy equivalence formula $E = mc^2$ has been dubbed "the world's most famous equation"

This expression is Einstein’s famous mass-energy equivalence equation, which shows that (speed-dependent) mass is a measure of the total energy of all forms. This equation applies to microscopic objects (single particles) as well as to macroscopic objects.
Energy Units

1 keV = 10^3 eV

1 MeV = 10^6 eV

1 GeV = 10^9 eV

0.013 keV

0.511 MeV/c^2

0.938 GeV/c^2

where c is the speed of light

Ionization energy of the hydrogen atom

Electron mass

Proton mass

Mass:

Proton = 1.6726231*10^{-27} kg

Electron = 9.1093897*10^{-31} kg

E = \gamma m_0 c^2

E = E_{kin} + m_0 c^2
Newton’s second law’s consequence:

\[ F = m \frac{dv}{dt} = ma, \]

- no upper limit on velocity

- Measured!

Relativity:

\[ E_{\text{kin}} = \gamma m_0 c^2 - m_0 c^2 \]

Newtonian prediction

\[ E_{\text{kin}} = \frac{mv^2}{2} \]

\[ \frac{1}{m_0 c^2} \left( E_{\text{kin}} + m_0 c^2 \right) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ 1 - \left( \frac{m_0 c^2}{E_{\text{kin}} + m_0 c^2} \right)^2 = \frac{v^2}{c^2} \]

\[ E_{\text{kin}} \gg m_0 c^2 \Rightarrow \frac{v^2}{c^2} = 1 \Rightarrow v^2 = c^2 \]
Example 1:
The proton’s rest mass is 0.938 GeV/c\(^2\). The total energy of a proton is three times its rest energy.

a) Find the proton’s rest energy

\[ E_{\text{rest}} = m_0 c^2 = 0.938 \, \text{GeV/c}^2 = 0.938 \, \text{GeV} \]

b) With what speed is the proton moving? Give your answer in [m/s] units.

\[ 1 - \frac{v^2}{c^2} = 3 \implies 3 \sqrt{1 - \frac{v^2}{c^2}} = 1 \implies 8 = \frac{9v^2}{c^2} \implies v = c \sqrt{\frac{8}{9}} \approx 2.83 \times 10^8 \frac{m}{s} \]
Example 2
An electron has a speed $v = 0.850c$. Its rest mass is 0.511 MeV/c$^2$. Find the electron’s:
a) total energy and kinetic energy.
b) Compare your result with the result from a classical calculation of kinetic energy. How large is the difference (quantify!)?

This is a relativistic problem. Do NOT use non-relativistic expression for $E_{\text{kin}}$

$$E = \gamma m_0 c^2$$

$$E = E_{\text{kin}} + m_0 c^2$$

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \approx 1.9$$

$$E = \gamma m_0 c^2 = 1.9 \times 0.511 \text{ MeV} \approx 0.97 \text{ MeV}$$

$$E_{\text{kin}} = E - m_0 c^2 \approx 0.97 \text{ MeV} - 0.511 \text{ MeV} = 0.46 \text{ MeV}$$

Classically: $E_{\text{kin}} = \frac{1}{2} m_0 v^2 = \frac{1}{2} \frac{0.511 \text{ MeV}}{c^2} \times (0.85)^2 c^2 = 0.18 \text{ MeV}$

Experiment tells us differently!
In many situations, the momentum or energy of a particle is measured rather than its speed.

\[ \vec{p} = \gamma m_0 \vec{v} \]
\[ E = \gamma m_0 c^2 \]

\[ \Rightarrow \left( m_0 c^2 \right)^2 = E^2 - p^2 c^2 \]

Useful invariant (must have the same value in all reference frames)

**Energy-momentum momentum relation for photons:**

photons: particles with zero “rest” mass, speed = c

\[ E^2 = c^2 p^2 + 0 \Rightarrow E = cp \]
Electromagnetic waves

\[ E = pc \]

<table>
<thead>
<tr>
<th>Frequency, sec(^{-1})</th>
<th>Photon energy, eV</th>
<th>Wavelength, m</th>
<th>Speed (with error), (\times 10^8 m/sec)</th>
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<tr>
<td>4.7 (\times) 10(^7)</td>
<td>1.9 (\times) 10(^{-7})</td>
<td>6.4</td>
<td>2.9978 (\pm) 0.0003</td>
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<td>1.7 (\times) 10(^8)</td>
<td>7.0 (\times) 10(^{-7})</td>
<td>1.8</td>
<td>2.99795 (\pm) 0.00003</td>
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<tr>
<td>3.0 (\times) 10(^8)</td>
<td>1.2 (\times) 10(^{-6})</td>
<td>1.0</td>
<td>2.99792 (\pm) 0.00002</td>
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<tr>
<td>3.0 (\times) 10(^9)</td>
<td>1.2 (\times) 10(^{-5})</td>
<td>1.0 (\times) 10(^{-1})</td>
<td>2.99792 (\pm) 0.00009</td>
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<tr>
<td>2.4 (\times) 10(^{10})</td>
<td>1.0 (\times) 10(^{-4})</td>
<td>1.2 (\times) 10(^{-2})</td>
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<tr>
<td>7.2 (\times) 10(^{10})</td>
<td>3.0 (\times) 10(^{-4})</td>
<td>4.2 (\times) 10(^{-3})</td>
<td>2.997925 (\pm) 0.000001</td>
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<tr>
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<td>2.2</td>
<td>5.6 (\times) 10(^{-7})</td>
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<td>5.1 (\times) 10(^5)</td>
<td>2.5 (\times) 10(^{-12})</td>
<td>2.983 (\pm) 0.015</td>
</tr>
<tr>
<td>4.1 (\times) 10(^{22})</td>
<td>1.7 (\times) 10(^8)</td>
<td>7.3 (\times) 10(^{-15})</td>
<td>2.97   (\pm) 0.03</td>
</tr>
</tbody>
</table>
Conservation of Relativistic Momentum and Energy

**Example:**
The $\pi^+$ meson is a subatomic particle. It is observed to decay at rest into a muon $\mu^+$ lepton and a neutrino

$$\pi^+ \rightarrow \mu^+ + \nu_\mu$$

The neutrino mass is very small compared with the pion and muon mass, and can safely be neglected. Neutrino cannot be detected directly. The rest mass of the muon $\mu^+$ is 106 MeV/c$^2$ and its kinetic energy is measured to be 4.6 MeV. Find the mass of $\pi^+$. 
Energy conservation:

\[ E_{\pi^+} = E_{\mu^+} + E_{\nu_\mu} \]

\[ m_{0\pi^+} c^2 = \sqrt{\left[ m_{0\mu^+} c^2 \right]^2 + \left[ p_{\mu^+} c \right]^2} + p_{\nu_\mu} c \]

since \( m_{0\mu} \approx 0 \)

Momentum conservation:

\[ p_{\mu^+} = p_{\nu_\mu} \]

\[ \Rightarrow m_{0\pi^+} c^2 = \sqrt{\left[ m_{0\mu^+} c^2 \right]^2 + \left[ p_{\mu^+} c \right]^2} + p_{\mu^+} c \]

\[ \left[ p_{\mu^+} c \right]^2 = \left[ E_{\mu^+} \right]^2 - \left[ m_{0\mu^+} c^2 \right]^2 = \left[ E_{\text{kin}} + m_{0\mu^+} c^2 \right]^2 - \left[ m_{0\mu^+} c^2 \right]^2 = \left[ E_{\text{kin}} \right]^2 + 2E_{\text{kin}} m_{0\mu^+} c^2 \]

\[ m_{0\pi^+} c^2 \sim 111 \text{MeV} + 31 \text{MeV} \sim 140 \text{MeV} \]

\[ m_{0\pi^+} \sim 140 \text{MeV}/c^2 \]
Recipe for relating two description of the same point event in two frames $S$ and $S'$

$P$ has a position $(x, y, z, t)$ in the $S$ frame and position $(x', y', z', t')$ in the $S'$ frame. If one knows the position in the $S$ frame, one can calculate what position would an object have in the $S'$ frame (and vice versa)

$$
x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} = \gamma (x - vt)
$$

$$
y' = y
$$

$$
z' = z
$$

$$
t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} = \gamma \left( t - \frac{vx}{c^2} \right)
$$

where $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$
Lorentz Transformations: energy and momentum

Recipe for relating two description of the same point event in two frames S and S’

\[
\dot{V} = (V,0,0) \quad (p_x',p_y',p_z',E')
\]

\[
P \text{ or } (p_x,p_y,p_z,E)
\]

P has momentum and energy \((p_x,p_y,p_z,E)\) in the S frame and momentum and energy \((p_x',p_y',p_z',E')\) in the S’ frame. If one knows the momentum and energy in the S frame, one can calculate what momentum and energy would an object have in the S’ frame (and vice versa)

\[
p_x' = \frac{p_x - VE/c^2}{\sqrt{1-V^2/c^2}} = \gamma_V \left( p_x - VE/c^2 \right)
\]

\[
p_y' = p_y
\]

\[
p_z' = p_z
\]

\[
E' = \frac{E - Vp_x}{\sqrt{1-V^2/c^2}} = \gamma_V \left( E - Vp_x \right)
\]

where \(\gamma_V = \frac{1}{\sqrt{1-V^2/c^2}}\)
Lorentz Invariant: spacetime interval

Most general case:

$$s^2 = (ct)^2 - x^2 - y^2 - z^2 = (ct')^2 - x'^2 - y'^2 - z'^2$$

y=z=y'=z'=0 :

$$s^2 = (ct)^2 - x^2 = (ct')^2 - x'^2$$

$s^2$ invariant means it has the same value in S and S’ frame (i.e. it does not depend on the frame)
Lorentz Invariant: rest energy

Most general case:

\[ E_{\text{rest}}^2 = \left( m_0 c^2 \right)^2 = E^2 - (\vec{p} c)^2 = E^2 - p_x^2 c^2 - p_y^2 c^2 - p_z^2 c^2 \]

\[ = E'^2 - p'^2_x c^2 - p'^2_y c^2 - p'^2_z c^2 = E'^2 - (\vec{p}' c)^2 \]

If \( p_y = p_z = p'_y = p'_z = 0 \) then:

\[ E_{\text{rest}}^2 = \left( m_0 c^2 \right)^2 \]

\[ = E^2 - (\vec{p} c)^2 = E^2 - p_x^2 c^2 = E'^2 - p'^2_x c^2 = E'^2 - (\vec{p}' c)^2 \]

\((E_{\text{rest}})^2\) invariant means it has the same value in S and S' frame (i.e. it does not depend on the frame)