Lecture 24:

5. Nuclei, Models
   Collective motion: vibrations, rotations

6. Nuclear decays: $\gamma$ ($\alpha, \beta$ next lecture)
Incomplete subshells, residual forces
(Beyond Shell Model)

Incomplete subshell: the states it can form with k nucleons is degenerate (i.e. has fixed energy). The presence of residual forces among nucleons separates states in the energy (degeneracy removed). The angular momentum of each state = result of adding k angular momenta $j$

Example: k “last” protons reside in a subshell $nlj$

\[
(n_1 l_1 j_1)^{2j_1+1} (n_2 l_2 j_2)^{2j_2+1} \ldots (nlj)^k
\]

If we don’t consider residual interaction, the $2j+1$ states which compose the last level (with k “valence” protons) are degenerate. The presence of interactions removes the degeneracy.

What combinations of the states are anti-symmetric with respect to exchange of the fermions, and therefore allowed? What is their combined net angular momentum?
Residual Interactions

\[ H = H_0 + H_{\text{residual}} \]

\( H_{\text{residual}} \) reflects interactions not in the single particle potential.

NOT a minor perturbation (assumed before)

Start with 2- particle system, that is, a nucleus “doubly magic + 2”.

\( H_{\text{residual}} \) is \( H_{12}(r_{12}) \)

Consider two identical valence nucleons with \( j_1 \) and \( j_2 \).

What total angular momenta \( j_1 + j_2 = j \) can be formed?
Example:
Nuclei with **two “valence” particles** outside doubly magic core. Universal result: $J = 0, 2, 4, 6 \ldots (2j-1)$.

Why these combinations?
Residual Interactions

\[ H = H_0 + H_{\text{residual}} \]

\( H_{\text{residual}} \) reflects interactions not in the single particle potential.

NOT a minor perturbation (assumed before)

Start with 2- particle system, that is, a nucleus “doubly magic + 2”.

\( H_{\text{residual}} \) is \( H_{12}(r_{12}) \)

Consider two identical valence nucleons with \( j_1 \) and \( j_2 \).

What total angular momenta \( j_1 + j_2 = j \) can be formed?

\[ j_1 + j_2 \quad \text{All values from:} \quad j_1 - j_2 \quad \text{to} \quad j_1 + j_2 \quad (j_1 \neq j_2) \]

Example: \( j_1 = 3, j_2 = 5 \): \( J = 2, 3, 4, 5, 6, 7, 8 \)

BUT: For \( j_1 = j_2 = j \): \( J = 0, 2, 4, 6, \ldots (2j - 1) \) Why?
How can we know which total J values are obtained for the coupling of two identical nucleons in the same orbit with total angular momentum j? “m-scheme” method.

<table>
<thead>
<tr>
<th>$j_1=7/2$</th>
<th>$j_2=7/2$</th>
<th>$M$</th>
<th>$J$</th>
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<tbody>
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</table>

*Only positive total M values are shown, the table is symmetric for M < 0.

R. F. Casten “Nuclear Structure from a Simple Perspective”
How can we know which total J values are obtained for the coupling of three identical nucleons in the same orbit with total angular momentum j? “m-scheme” method.

<table>
<thead>
<tr>
<th>( j_1 = 5/2 )</th>
<th>( j_2 = 5/2 )</th>
<th>( j_3 = 5/2 )</th>
<th>M</th>
<th>J</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_1 )</td>
<td>( m_2 )</td>
<td>( m_3 )</td>
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<td>9/2</td>
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<td>1/2</td>
<td>-3/2</td>
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</tbody>
</table>

*The full set of allowable \( m_i \) combinations that give \( M > 0 \) are obtained by the conditions \( m_1 > 0, m_3 < m_2 < m_1 \) and no two \( m_i \) values identical.

R. F. Casten “Nuclear Structure from a Simple Perspective”
How can we know which total J values are obtained for the coupling of three identical nucleons in the same orbit with total angular momentum j? “m-scheme” method.

<table>
<thead>
<tr>
<th>$M_F^{(3)}$</th>
<th>Possible states $m_J^{(1)} m_J^{(2)} m_J^{(3)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>9/2</td>
<td>$\begin{pmatrix} 5 &amp; 3 &amp; 1 \ 2 &amp; 2 &amp; 2 \end{pmatrix}$</td>
</tr>
<tr>
<td>7/2</td>
<td>$\begin{pmatrix} 5 &amp; 3 &amp; 1 \ 2 &amp; 2 &amp; 2 \end{pmatrix}$</td>
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<tr>
<td>5/2</td>
<td>$\begin{pmatrix} 5 &amp; 1 &amp; 1 \ 2 &amp; 2 &amp; 2 \end{pmatrix}$, $\begin{pmatrix} 5 &amp; 3 &amp; -3 \ 2 &amp; 2 &amp; 2 \end{pmatrix}$</td>
</tr>
<tr>
<td>3/2</td>
<td>$\begin{pmatrix} 5 &amp; 1 &amp; 3 \ 2 &amp; 2 &amp; 2 \end{pmatrix}$, $\begin{pmatrix} 5 &amp; 3 &amp; -5 \ 2 &amp; 2 &amp; 2 \end{pmatrix}$, $\begin{pmatrix} 3 &amp; 1 &amp; -1 \ 2 &amp; 2 &amp; 2 \end{pmatrix}$</td>
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<td>1/2</td>
<td>$\begin{pmatrix} 5 &amp; -1 &amp; 3 \ 2 &amp; 2 &amp; 2 \end{pmatrix}$, $\begin{pmatrix} 3 &amp; 1 &amp; 3 \ 2 &amp; 2 &amp; 2 \end{pmatrix}$, $\begin{pmatrix} -3 &amp; 1 &amp; -1 \ 2 &amp; 2 &amp; 2 \end{pmatrix}$</td>
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<td>$\begin{pmatrix} -5 &amp; -1 &amp; 3 \ 2 &amp; 2 &amp; 2 \end{pmatrix}$, $\begin{pmatrix} -3 &amp; -1 &amp; 3 \ 2 &amp; 2 &amp; 2 \end{pmatrix}$, $\begin{pmatrix} 5 &amp; 1 &amp; 3 \ 2 &amp; 2 &amp; 2 \end{pmatrix}$</td>
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<td>$\begin{pmatrix} -5 &amp; -1 &amp; 1 \ 2 &amp; 2 &amp; 2 \end{pmatrix}$, $\begin{pmatrix} -5 &amp; -3 &amp; 3 \ 2 &amp; 2 &amp; 2 \end{pmatrix}$</td>
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<td>$\begin{pmatrix} -5 &amp; -3 &amp; -1 \ 2 &amp; 2 &amp; 2 \end{pmatrix}$</td>
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</table>

J=5/2, (2J+1)=6 different one-particle states

3 5/2 particles:

6!/3!(6-3)!=20 different anti-symmetric states for 3 particles

http://www.eng.fsu.edu/~dommelen/quantum/style_a/nt_shell.html#sec:nt_shell
Possible combined angular momentum of identical fermions in shells of single-particle states that differ in magnetic quantum number. The top shows odd numbers of particles, the bottom even numbers.

### Extra Lecture 23

| $j^p$ | $I$ | $\frac{1}{2}$ | $\frac{3}{2}$ | $\frac{5}{2}$ | $\frac{7}{2}$ | $\frac{9}{2}$ | $\frac{11}{2}$ | $\frac{13}{2}$ | $\frac{15}{2}$ | $\frac{17}{2}$ | $\frac{19}{2}$ | $\frac{21}{2}$ | $\frac{23}{2}$ | $\frac{25}{2}$ | $\frac{27}{2}$ | $\frac{29}{2}$ | $\frac{31}{2}$ | $\frac{33}{2}$ | $\frac{35}{2}$ |
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| $\frac{1}{2}$ | 1   | 1             |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |
| $\frac{3}{2}$ | 1   | 1             |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |
| $\frac{5}{2}$ | 1   |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |
|         | 3   | 1             | 1             |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |
| $\frac{7}{2}$ | 1   |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |
|         | 3   | 1             | 1             | 1             | 1             |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |
| $\frac{9}{2}$ | 1   |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |
|         | 3   | 1             | 1             | 1             | 1             | 1             |               |               |               |               |               |               |               |               |               |               |               |               |               |               |
|         | 5   | 1             | 1             | 2             | 2             | 2             | 2             | 2             | 2             | 2             | 1             | 1             |               |               |               |               |               |               |               |               |
| $\frac{11}{2}$ | 1   |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |
|         | 3   | 1             | 1             | 1             | 2             | 2             | 1             | 1             | 1             | 1             |               |               |               |               |               |               |               |               |               |               |
|         | 5   | 1             | 2             | 3             | 4             | 4             | 4             | 4             | 4             | 3             | 3             | 2             | 2             | 2             | 2             | 1             | 1             |               |               |

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</table>
**Lecture 23**

**Incomplete subshells, residual forces**

*Beyond Shell Model*

Total angular momenta of \( k \) identical nucleons placed in a sub-shell of angular momentum \( j \):

<table>
<thead>
<tr>
<th>( j )</th>
<th>( k )</th>
<th>( J )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>1</td>
<td>1/2</td>
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<td></td>
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<td>3/2,5/2,9/2</td>
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</tbody>
</table>

Several configurations, restrictions imposed by the anti-symmetrization for given values of the total angular momentum.

Different total angular momenta, degeneracy removed (residual multi-nucleon interaction in the shell \( nlj \))
Incomplete subshells, residual forces
(Beyond Shell Model)

Total angular momenta of $k$ identical nucleons placed in a sub-shell of angular momentum $j$:

<table>
<thead>
<tr>
<th>$j$</th>
<th>$k$</th>
<th>$J$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/2$</td>
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<td>$3/2,5/2,9/2$</td>
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</tbody>
</table>

Several configurations, restrictions imposed by the anti-symmetrization for given values of the total angular momentum.

$^{11}\text{Na}^{23}$

predicted measured
Even-Even nuclei and Collective Structure
Even-Even nuclei and Collective Structure

Sn (Tin) Z=50, N=80

Ground state
Sn (Tin) $Z=50$, $N=80$

Ground state

Excited state:
breaking a pair (~2MeV)

What’s spin and parity of this state?
Even-Even nuclei and Collective Structure

Sn (Tin) Z=50, N=80

Ground state

$E$ (MeV)

$1_{1/2}$ 3s1/2

Excited state: breaking a pair (~2MeV)

$P \left( s_{1/2} \otimes h_{11/2} \right) = P \left( s_{1/2} \right) \times P \left( h_{11/2} \right) = (+1) \times (-1) = -1$
Even-Even nuclei and Collective Structure

Sn (Tin) Z=50, N=80

Ground state

Expected (excited) shell model states

1h9/2

n: 1h11/2
3s1/2

\[
\begin{align*}
|J| &= \left| s_{1/2} - h_{9/2} \right|, \ldots, \left| s_{1/2} + h_{9/2} \right| = 4,5 \\
P(s_{1/2} \otimes h_{9/2}) &= P(s_{1/2}) \times P(h_{9/2}) = (+1) \times (-1) = -1 \\
|J| &= \left| h_{11/2} - h_{9/2} \right|, \ldots, \left| h_{11/2} + h_{9/2} \right| = 1,2,3,4,5 \\
P(h_{11/2} \otimes h_{9/2}) &= P(h_{11/2}) \times P(h_{9/2}) = (-1) \times (-1) = +1
\end{align*}
\]
Experimental observation:
Hundreds of known even-even nuclei in the shell model region each one has an anomalous $2^+$ (lowest excited) state. This is a general property of even-even nuclei.

What are $2^+$ states below $E=2$ MeV?
- not shell model states
- new states, result of nuclear collective motion
  - $A<150$ vibrations
  - $150 < A < 190$ and $A > 230$ rotations

The collective nuclear model
= the liquid drop model
Experimental observation:
Hundreds of known even-even nuclei in the shell model region each one has an anomalous $2^+$ (lowest excited) state. This is a general property of even-even nuclei.

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The collective nuclear model
= the liquid drop model
Collective excitations modes involve all the nucleons in coherent motion.

Assumption: nuclear surface can accomplish oscillations around an equilibrium.
Nuclear Vibrations

Assumption: nuclear surface can accomplish oscillations around an equilibrium.

Parameterization of the nuclear surface (spherical harmonics expansion)

\[ R(t) = R_{av} + \sum_{\lambda \geq 1} \sum_{\mu = -\lambda}^{+\lambda} \alpha_{\lambda \mu}(t) Y_{\lambda \mu}(\theta, \phi) \]

Average shape is spherical, the instantaneous shape is not.

Main vibrational modes:

- Isoscalar dipole vibrations (\(\lambda=1\)) shifts of the center of mass (not allowed in the absence of external forces)
- Quadrupole vibrations (\(\lambda=2\)) - the lowest occurring mode
Phonons: Quanta of Vibrational Energy

Mechanical vibrations = production of vibrational phonons

E.g. Tellurium $^{120}_{52}$Te

Vibrational band

- 3 phonon states
- 2 phonon states (triplet)
- 1 phonon state (a phonon carries 2 units of angular momentum)

Ground state

If the vibration is harmonic, the states are equidistant
**Phonons: Quanta of Vibrational Energy**

**Quadrupole vibrations \( (\lambda = 2) \)**

\( \lambda = 2 \) phonon carries:

- 2 units of angular momentum
  - It adds a \( Y_{2\mu} \) dependence to the nuclear wave function and even parity
  - Parity of \( Y_{1m} \) is \((-1)^l\)

Since it creates a shape which has \( \psi(r) = \psi(-r) \). The energy of the quadrupole phonon is not predicted by this theory (adjustable parameter).

An even-even nucleus in the ground state has \((-1)^\pi = 0^+\)

- Adding a \( \lambda = 2 \) phonon creates the first excited state \( I^\pi = 2^+ \).
- Adding two \( \lambda = 2 \) phonons gives \( I^\pi = 0^+, 2^+, 4^+ \) (triplet state)
- Adding three \( \lambda = 2 \) phonons gives \( I^\pi = 0^+, 2^+, 3^+, 4^+, 6^+ \).

<table>
<thead>
<tr>
<th>two ( \lambda = 2 ) phonons</th>
<th>( \mu_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu_2 )</td>
<td>-2</td>
</tr>
<tr>
<td>-2</td>
<td>-4</td>
</tr>
<tr>
<td>-1</td>
<td>-3</td>
</tr>
<tr>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>+1</td>
<td>-1</td>
</tr>
<tr>
<td>+2</td>
<td>0</td>
</tr>
</tbody>
</table>

The entries show \( \mu = \mu_1 + \mu_2 \).

\( l = 4 \)  \( \mu = +4, +3, +2, +1, 0, -1, -2, -3, -4 \)
\( l = 2 \)  \( \mu = +2, +1, 0, -1, -2 \)
\( l = 0 \)  \( \mu = 0 \)

Triplet of states with spins 0+, 2+, 4+ at twice the energy of the first 2+ state (2 identical phonons carry twice as much energy as one)
Vibrational Model

Predictions:

- If the equilibrium shape is spherical, the quadrupole moments of the 2+ should vanish
- The predicted ratio of $E(4+)/E(2+)$ ~2.0

These predictions work well for $A<150$. 

![Graphs showing energy levels and quadrupole moments](image)
Nuclear Rotations

Rotational motion can be observed only in nuclei with non-spherical equilibrium shapes. These nuclei can have substantial distortions from spherical shape (deformed nuclei in the ground state: 150<A<190 rare earths and A>220 actinides).

\[ R(\theta, \varphi) = R_{\text{ave}} \left[ 1 + \beta Y_{20}(\theta, \varphi) \right] \]

\[ \beta = \frac{4}{3} \sqrt{\frac{\pi}{5}} \frac{\Delta R}{R_{\text{ave}}} \]

Deformation parameter

\[ R_{\text{ave}} = R_0 A^{1/3} \]

If nuclei can be deformed they may be able to rotate

\[ \beta > 0 \quad \text{Prolate} \]

\[ \beta < 0 \quad \text{Oblate} \]
Rotational Energy and Spectrum of Rotational States

- **Rigid classical rotator:**
  \[ E = \frac{1}{2} J \omega^2 \]
  \[ I = J \omega \]
  \[ \Rightarrow E = \frac{1}{2} \frac{I^2}{J} \]
  where \( J \) – moment of inertia (a measure of object resistance to changes to its rotation) [kg m\(^2\)]

- **Rigid Quantum Mechanical rotator:**
  \[ E = \frac{1}{2} \frac{I(I + 1) \hbar^2}{J} \]
  where \( I \) – angular momentum quantum number

- Increasing the quantum number \( I \) = adding rotational energy to the nucleus.
- Nuclear excited states form a sequence known as a rotational band.
Rotational Energy and Spectrum of Rotational States

- The ground state of even-even nuclei is $0^+$
- The mirror symmetry of the nucleus $\psi(-\vec{r}) = \psi(\vec{r})$ restricts the sequence of rotational states to even values of $I$ and positive parity:

$$E = \frac{1}{2} \frac{I(I+1)\hbar^2}{J}$$

$E = 42 \frac{\hbar^2}{2j}$  $6^+$

$E = 20 \frac{\hbar^2}{2j}$  $4^+$

$E = 6 \frac{\hbar^2}{2j}$  $2^+$

$0^+$

→ Obtain moment of inertia
Rotational Spectrum

\[ E(2^+) = \frac{6\hbar^2}{2J} = 91.4\text{keV} \Rightarrow \frac{\hbar^2}{2J} = 15.2\text{keV} \]

\[ E(4^+) = \frac{20\hbar^2}{2J} = 20 \times 15.2\text{keV} = 304\text{keV} \]

e tc

\[ I \quad \text{Energy (keV)} \]

\[ 12^+ \quad 2082.7 \]

\[ 10^+ \quad 1518.1 \]

\[ 8^+ \quad 1024.6 \]

\[ 6^+ \quad 614.4 \]

\[ 4^+ \quad 299.5 \]

\[ 2^+ \quad 91.4 \]

\[ 0^+ \quad 0 \]

**Figure 5.22** The excited states resulting from rotation of the ground state in $^{164}$Er. (typical rotational nucleus)
Er=Erbium (Z= 68)
**Moment of Inertia of Nuclei**

\[ J_{\text{fluid}} < J_{\text{observed}} < J_{\text{rigid}} \]

\[ \frac{\hbar^2}{2J_{\text{fluid}}} \approx 90 \text{keV} \quad \frac{\hbar^2}{2J_{\text{observed}}} \approx 15 \text{keV} \quad \frac{\hbar^2}{2J_{\text{rigid}}} \approx 6 \text{keV} \]

- Nucleus does not show the long range structure that would increase the solid rigid
- Increase of moment of inertia that occurs at high angular momentum or rotational frequency (this effect is called “centrifugal stretching”)
- Nucleon’s rotational motion slower than internal motion

\[ E_{\text{kin}} \text{ (nucleon)} \sim 20 \text{ MeV} \quad \omega = \frac{\sqrt{2E}}{J} \quad v=0.2c \]

**The picture of a rotating deformed nucleus:** A stable equilibrium shape determined by nucleons in rapid internal motion in the nuclear potential, with the entire resulting distribution rotating slowly that the rotation has little effect on the nuclear structure or on nuclear orbits.
Figure 5.23  The states of $^{164}$Er below 2 MeV. Most of the states are identified with three rotational bands: one built on the deformed ground state, a second built on a $\gamma$-type vibration (in which the surface vibrates transverse to the symmetry axis), and a third built on a $\beta$-type vibration (in which the surface vibrates along the symmetry axis). Many of the other excited states originate from pair-breaking particle excitations and their associated rotational bands.
Nillson Model

Central potential no longer spherically symmetric

\[ V_{\text{Nillson}} = \frac{1}{2} m \left[ \omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2 \right] + V_{ls} l \cdot s + V_{ll} l^2 \] (Single particle)

- Use 3-dimentional deformed harmonic oscillator potential with axial symmetry and with spin-orbit and \( l^2 \) terms.
- Solve the Schrodinger equation to get the energy levels.

Assuming axial symmetry:

\[ \omega_\perp^2 \equiv \omega_x^2 = \omega_y^2 = \omega_0^2 (1 + \frac{2}{3} \delta) \]

\[ \omega_z^2 = \omega_0^2 (1 - \frac{4}{3} \delta) \]

 Frequencies are functions of 2 parameters, which can related through the condition imposed by the conservation of nuclear volume:

\[ \omega_x \omega_y \omega_z = \text{const} = \bar{\omega}_0^3 \] (frequency for a zero deformation)

\[ \omega_0 = \bar{\omega}_0 \left( 1 - \frac{4}{3} \delta^2 - \frac{16}{27} \delta^3 \right)^{-1/6} \]

\[ \hbar \bar{\omega}_0 = 41 \text{MeV} \times A^{-1/3} \]

Modified energy levels
The Deformed Shell Model
Nillson Diagram

Extra
Microscopic theories of transitions

Multipolarity of transition \( J_i^{\pi} \rightarrow J_j^{\pi} \) is \( \Delta J^{\Delta \pi} \)

\[ |J_i - J_j| \leq \Delta J \leq J_i + J_j \text{ and } \Delta \pi = \pi_i \pi_j \]

Electromagnetic Transitions

- provide the most elegant way to collective excitations in nuclei.

Types of electromagnetic transitions:

- Electric dipole transition
- Higher multipoles (magnetic dipole, electric quadrupole etc)
Electric Dipole Transitions (E1) (photons)

Classical example: Herz dipole
- the power output emitted by the dipole \( \sim \omega^4 \).

Transition probability = the rate of photon emission = power output and photon energy ratio:

\[
W_{fi} = \frac{1}{\tau} = \frac{e^2}{3\pi \epsilon_0 \hbar^4 c^3} E_\gamma^3 \left| \int d^3 x \psi_f^* x \psi_i \right|^2 \quad \text{(Quantum Mechanics)}
\]

- Electric dipole (E1) transitions always connect states with different parity.
- Photon carries angular momentum \( |l| = 1\hbar \Rightarrow \) the angular momenta of the initial and final states may differ by one unit.

The dipole approximation crudely restricts consideration to the case where the emitted photon can only have zero orbital angular momentum.
Electric Quadrupole Transitions (E2) (photons)

Electric quadrupole (E2) transition probability:

\[
W_{fi} = \frac{1}{\tau} = \frac{e^2}{3\pi\varepsilon_0 h^4 c^3} E_\gamma^5 \left| \int d^3 x \psi_f^* x \psi_i \right|^2
\]  

(Quantum Mechanics)

- Electric quadrupole (E2) transitions always connect states with the same parity.
- The angular momenta of the initial and final states may differ by two units: \( |j_f - j_i| \leq 2 \leq j_f + j_i \)

These are so called “selection rules”

Examples: \( \gamma \) decay and strength of a transition.

Labeling of transitions \( \Delta J^\pi \) (gamma decay):

- If \( \pi = (-1)^{\Delta J} \) \( \Rightarrow \) E”J” transition (electric): \( E1=1^- \), \( E2=2^+ \), \( E3=3^- \)
- If \( \pi = -(-1)^{\Delta J} \) \( \Rightarrow \) M”J” transition (magnetic): \( M1=1^+ \), \( M2=2^- \), \( M3=3^+ \)
Giant Resonance

- the observed concentration of excitation strength at $E \sim 10$’s of MeV. They originate from collective excitation of nucleons.

The total value and distribution width much larger than typical resonances built upon single-particle (non-collective) excitations.

For most giant resonances, the strength is found to be $\sim$ independent of the probe used to excite the nucleus ($\gamma$-rays, e, p, $\alpha$, heavy-ions)

Both the width and peak of strength distribution vary smoothly with $A$: e.g. isovector giant dipole resonance:

$$E_1 \sim 78 A^{-1/3} [\text{MeV}]$$
Giant Dipole (E1) Resonance

- Protons and neutrons act as two separate group of particles.
- Excitation comes from the motion of one group with respect to the other (little or no excitation within each group)

Isovector Dipole mode:

Protons and neutrons are moving in opposite directions (the opposite phase keeps the center of mass of the nucleus stationary) or “out of phase” with respect to each other → isovector mode of excitation.

Note: if protons and neutrons are moving in phase, it is an isoscalar dipole vibration (entire nucleus is oscillating around some equilibrium position in the lab). This is of no interest if we wish to study internal dynamics of a nucleus.
Giant Dipole (E1) Resonance

\[ \Delta J^{\Delta \pi} = 1^- \] nucleons are promoted up one major shell (particle-hole)

The energy gap between two adjacent major shells: \[ \Delta E \sim 41A^{-1/3} \text{[MeV]} \]

The location of the peaks of strength: \[ E_1 \sim 78A^{-1/3} \text{[MeV]} \]

\[ E_1 > \Delta E \]

Note: the difference is caused by residual interactions between nucleons, which pushes isovector excitations to higher energies.

The explanation of giant dipole resonance is provided by the Goldhaber-Teller model, based on collective motion of nucleons.
Giant Resonance

Figure 6-1: Time evolution of low-order vibrational modes. The monopole oscillation in (a) involves variations in the size without changing the overall shape. The nucleus moves as a whole in an isoscalar dipole vibration shown in (b). In contrast, an isovector dipole vibration consists of neutrons and protons oscillating in opposite phase, as in (c). In quadrupole vibrations the nucleus changes from prolate to oblate and back again, as in (d). Octupole vibrations are shown in (e).
\textbf{Giant Resonances}

\[ ^{90}\text{Zr} \]

\textbf{Zirconium}

\[ ^{90}\text{Zr} \, (p,p') \]
\[ E_p = 200 \text{ MeV} \]

4 deg

\[ E1 \]

8 deg

\[ E2 \]

\[ \Delta E \, (\text{FWHM}) \approx 1 \text{ MeV} \]

Bertrand et al. (1981)
Giant Resonances

\(^{90}\text{Zr}\)

Zirconium
Z = 40

\[ N = 5 \]
\[ N = 4 \]
\[ N = 3 \]

Dipole \( J^\pi = 1^- \)
Quadrupole \( J^\pi = 2^+ \)

Giant vibration \( \hat{\pi} \) = coherent superposition of elementary p-h excitations

A. Richter
The Valley of Stability

The largest numbers of stable nuclei are near “magic numbers”
**Nuclear Decays**

- A nucleus in an excited state is unstable, it can always undergo a transition to a lower-energy state of the same nucleus (emission of $\gamma$ radiation).

- A nucleus in an excited state or a ground state can undergo a transition to a lower-energy state of the another nucleus (emission of $\alpha$, $\beta$ radiation with or without subsequent $\gamma$ emission).
Electromagnetic Decays
Gamma decay of bismuth-214.

The daughter isotope is a more stable (lower-energy) version of the original bismuth-214.
$^{177}\text{Hf}$ (Hafnium)

The energy state levels are:

The gamma spectrum observed:

The energy differences between levels can be deduced from the energy spectrum of the photons seen.
Electromagnetic Decays

- When a heavy (parent) nucleus disintegrates by $\alpha, \beta$ decay or by fission, the daughter nucleus is often left in an excited state.

- If this state is below the excitation energy for fission, it will de-excite (e-m radiation)

- E-m radiation can be described in a series expansion as a superposition of different photon multipolarities:

  - **electric multipolarity**: $E_l$ (E1,E2,E3 etc) 
    parity = $(-1)^l$

  - **magnetic multipolarity**: $M_l$ (M1,M2,M3 etc) 
    parity = $(-1)^{l+1}$

  \[ J_i \rightarrow J_f : \quad |J_i - J_f| \leq l \leq J_i + J_f \]

  $l$ – photon(s) angular momentum

---

States with excitation energies > 8 MeV (binding energy per nucleon) can emit single nucleons

---

Fig. 3.10. Sketch of typical nuclear energy levels. The example shows an even-even nucleus whose ground state has the quantum numbers $0^+$. To the left the total cross-section for the reaction of the nucleus $^{A-1}X$ with neutrons (elastic scattering, inelastic scattering, capture) is shown; to the right the total cross-section for $\gamma$-induced neutron emission $^{A}X + \gamma \rightarrow ^{A-1}X + n$. 

---

58
For $\gamma$-ray emission from nuclei, the transition rate $\lambda$ depends on
– the type of radiation mechanism,
– the angular momentum carried away by the photon

For a transition from a state with angular momentum quantum number $j_i$ to a state with angular momentum quantum number $j_f$.

The photon carries angular momentum $\vec{L}$: $\vec{J}_f = \vec{J}_i + \vec{L}$

The photon angular momentum quantum number $l$ is:

$$|j_i - j_f| \leq l \leq j_i + j_f$$
Electromagnetic Transitions

Parity: \( \pi_i = \pi_f \pi_\gamma \)

Electric multipole transitions: \( \pi_i \pi_f = (-1)^l \)

Magnetic multipole transitions: \( \pi_i \pi_f = (-1)^{l+1} \)

Selection rules:

<table>
<thead>
<tr>
<th>( l )</th>
<th>( \pi_\gamma )</th>
<th>Radiation type</th>
<th>Label</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>Electric Dipole radiation</td>
<td>E1</td>
</tr>
<tr>
<td>1</td>
<td>+1</td>
<td>Magnetic Dipole radiation</td>
<td>M1</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>Magnetic Quadrupole radiation</td>
<td>M2</td>
</tr>
<tr>
<td>2</td>
<td>+1</td>
<td>Electric Quadrupole radiation</td>
<td>E2</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td>Electric Octapole radiation</td>
<td>E3</td>
</tr>
<tr>
<td>3</td>
<td>+1</td>
<td>Magnetic Octapole radiation</td>
<td>M3</td>
</tr>
</tbody>
</table>
Electromagnetic Transitions

Example:

\[ \begin{align*}
\pi_\gamma &= \pi_i \pi_f = (-1)(+1) = -1 \\
|2 - 1| &\leq l \leq 2 + 1
\end{align*} \]

Therefore \( l^{\pi} = 1^-, 2^-, \) or \( 3^- \)

i.e. we can have an E1, M2, or E3 transition.

In general:

- Lowest order is always the most probable (largest \( \lambda \) constant).
- Electric transitions are more probable than magnetic transitions.
Electromagnetic Decays

\[ ^{18}Ar^{38} \]

\[ 3^- \rightarrow 2^+ + \gamma \]
\[ |\Delta J| = 1(E1, dipol) \]
\[ 2^+ \rightarrow 0^+ + \gamma \]
\[ |\Delta J| = 2(E2, quadrupol) \]

Lowest possible I
Electromagnetic Decays

Table 3.1. Selection rules for some electromagnetic transitions

<table>
<thead>
<tr>
<th>Multipolarity</th>
<th>Electric</th>
<th>Magnetic</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>E\ell</td>
<td>M\ell</td>
</tr>
<tr>
<td>Dipole</td>
<td>E1 1 -</td>
<td>M1 1 +</td>
</tr>
<tr>
<td>Quadrupole</td>
<td>E2 2 +</td>
<td>M2 2 -</td>
</tr>
<tr>
<td>Octupole</td>
<td>E3 3 -</td>
<td>M3 3 +</td>
</tr>
</tbody>
</table>

- The lower the multipolarity, the larger the transition probability
- In a series of excited states $0^+, 2^+, 4^+$, the most probable decay is by a cascade of E2 transitions $4^+ \rightarrow 2^+ \rightarrow 0^+$ and not by a single $4^+ \rightarrow 0^+$ E4 transition
- Magnetic transitions are weaker than electric transitions of the same multipolarity
Electromagnetic Decays

- When a heavy (parent) nucleus disintegrates by $\alpha$, $\beta$ decay or by fission, the daughter nucleus is often left in an excited state.

- If this state is below the excitation energy for fission, it will de-excite (e-m radiation)

- E-m radiation can be described in a series expansion as a superposition of different photon multipolarities:

  - electric multipolarity: $E_l$ (E1,E2,E3 etc)
    parity = $(-1)^l$

  - magnetic multipolarity $M_l$ (M1,M2,M3 etc)
    parity = $(-1)^{l+1}$

  \[ J_i \rightarrow J_f : \quad |J_i - J_f| \leq l \leq J_i + J_f \]

  $l$ – photon(s) angular momentum

- the lifetime of a state (and a decay probability) strongly depends upon the multipolarity of the $\gamma$ transitions.

  \[
  \text{decay probability } \sim \left(E_\gamma\right)^{2l+1} B^{E,M}(l)
  \]

(Wong’s text book Sec. 5.3, 5.4)