Lecture 22:
5. Nuclei:
   - Nuclear models:
     ✓ Fermi gas model (Povh, ch 17.1)
     ✓ Shell model (Povh, ch 17.3)

and Wong!
The Fermi Gas Model

Assumption: nucleons move almost freely inside the nucleus (no scattering, consequence of Pauli principle)

Schrodinger equation:

\[-\frac{\hbar^2}{2m} \nabla^2 \psi = E \psi\]

free particle (V=0)
m, E – nucleon mass and energy

Simplification: Cartesian coordinates (use cube instead of sphere)

Boundary conditions: \(\psi(x, y, z) = 0\) for \(x = y = z = 0\) and \(x = y = z = a\)

Solution:

\[\psi(x, y, z) = A \sin(k_x x) \sin(k_y y) \sin(k_z z)\]

\[k_x a = n_x \pi, \quad k_y a = n_y \pi, \quad k_z a = n_z \pi\]

\[E(n_x, n_y, n_z) = \frac{\hbar^2 k_x^2}{2m} = \frac{\hbar^2}{2m} \left( k_x^2 + k_y^2 + k_z^2 \right) = \frac{\hbar^2 \pi^2}{2ma^2} n^2\]

\[n^2 = n_x^2 + n_y^2 + n_z^2\] “Quantization of a particle in a box”
The possible number of solutions with k between k and dk:

\[ dn(k) = \frac{1}{8} \frac{4\pi k^2 dk}{(\pi/a)^3} \Rightarrow dn(E) = \frac{\sqrt{2} m^{3/2} a^3}{2\pi^2 \hbar^3} E^{1/2} dE \]

\[ n(E_F) = \int_0^{E_F} dn(E) = \frac{\sqrt{2} m^{3/2} a^3}{3\pi^2 \hbar^3} E_F^{3/2} = \frac{A}{4} \]

\( E_F \) - Fermi energy, min. energy to include all nucleons
The Fermi Gas Model

\[ n(E_F) = \frac{\sqrt{2m^{3/2}a^3}}{3\pi^2\hbar^3} E_F^{3/2} = \frac{A}{4} \implies E_F = \frac{\hbar^2}{2m} \left( \frac{3\pi^2 \rho}{2} \right)^{2/3} \]
\[ \rho = \frac{A}{a^3} \]

If maximum energy is different for protons and neutrons, then:

\[ E_F(p) = \frac{\hbar^2}{2m} \left( \frac{3\pi^2 \rho_p}{2} \right)^{2/3} \quad \rho_p = \frac{Z}{a^3} \]
\[ E_F(n) = \frac{\hbar^2}{2m} \left( \frac{3\pi^2 \rho_n}{2} \right)^{2/3} \quad \rho_n = \frac{N}{a^3} \]

Number of nucleons with energy between \( E \) and \( E+dE \) at \( T=0 \) (nucleus ground state)

\[ T = 0 : n(E > E_F) = 0 \quad \text{(no particles with } E > E_F \text{)} \]
Exercise (home)

Using \( \rho = 0.17 \frac{nucleons}{fm^3} \) (density of a nucleus)

a) What is the Fermi energy?
   \( E_F \sim 37 \text{ MeV} \)

b) What is the Fermi momentum?
   \( p_F \sim 250 \text{ MeV/c} \)

(no Coulomb force)
Nuclear Binding Energy $E_B$

Weizacker semi-empirical mass formula (SEMF) 

based on analogy of a nucleus with a drop of incompressible fluid

$$E_B(Z, N) = \alpha_1 A - \alpha_2 A^{2/3} - \alpha_3 \frac{Z(Z-1)}{A^{1/3}} - \alpha_4 \frac{(N-Z)^2}{A} + \Delta$$

**Asymmetry term**

$$E_F(p) = C \left( \frac{Z}{A} \right)^{2/3}, E_F(n) = C \left( \frac{N}{A} \right)^{2/3}$$

$$E_T = \int_0^{E_F} E \, dn = \frac{3}{5} Z E_F(p)$$

$$= \frac{3}{5} N E_F(n)$$

$$E = C' A^{-2/3} \left( Z^{5/3} + N^{5/3} \right)$$

$$dn(E) = \frac{\sqrt{2m^{3/2}a^3}}{2\pi^2\hbar^3} E^{1/2} \, dE$$

$$\rho = \frac{A}{a^3}$$

$$E_F = \frac{\hbar^2}{2m} \left( \frac{3\pi^2 \rho}{2} \right)^{2/3}$$
$E = C'A^{-2/3} \left( Z^{5/3} + N^{5/3} \right)$

$E_{\text{min}} = C'A^{-2/3} \left( Z^{5/3} + N^{5/3} \right) \bigg|_{Z=N=A/2}$

$E - E_{\text{min}} = C'A^{-2/3} \left[ Z^{5/3} + N^{5/3} - 2 \left( A/2 \right)^{5/2} \right]$

$D = (N - Z) / 2 = N - A / 2 = A / 2 - Z$

$E - E_{\text{min}} = C'A^{-2/3} \left[ \left( A/2 + D \right)^{5/3} + \left( A/2 - D \right)^{5/3} - 2 \left( A/2 \right)^{5/3} \right]$

$f(x + a) = f(a) + xf'(a) + \frac{x^2}{2} f''(a) + ...$

$E - E_{\text{min}} = \frac{10}{9} C' \left( Z - A/2 \right)^2 + ...$

Asymmetry term in Liquid Drop Model

Imbalance between p and n increases the energy of the system (binding energy decreases)
The Fermi Gas Model

- Light nucleus: $Z \sim N$
- Heavy nucleus: $Z < N$
  - Proton well shallower than that for neutrons
- Even-even nuclei: abundant
- Odd-Odd nuclei: (almost) non-existent ($\beta$ emission)
Models of the Nuclei

- **Fermi gas model**
  
  → Describes large systems (quantization of angular momentum may be neglected), e.g. nucleons in neutron stars or electrons in white dwarfs.

- **Shell model – improvement upon the Fermi gas model** (realistic potential and spin-orbit interaction)
  
  → System of nucleons inside a nucleus is very small and posses discrete energy levels with distinct angular momenta.
  → *Nuclei shapes can be determined by nucleon-nucleon interaction.*
Magic Numbers

Configuration of a magic numbers of protons and neutrons is unusually stable (unsually abundant)

protons: 2, 8, 20, 28, 50, 82  
neutrons: 2, 8, 20, 28, 50, 82, 126, 184

http://hyperphysics.phy-astr.gsu.edu/hbase/nuclear/shell.html
**Magic Numbers**

Configuration of a magic numbers of protons and neutrons is unusually stable (unsually abundant)

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Helium - very tightly bound (doubly magic $\alpha$ particle)

$^5$He does not exist (an extra nucleon cannot be attached to the helium core)

**Magic Numbers**

Configuration of a magic numbers of protons and neutrons is unusually stable (unsually abundant)

**protons:** 2, 8, 20, 28, 50, 82  
**neutrons:** 2, 8, 20, 28, 50, 82, 126, 184

**“Magic nuclei”:**
- More stable isotops than other nuclei
- Small electric quadrupole moment

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\(^{16}\text{O}(Z=8,N=8)\) - very tightly bound  
\(^{17}\text{O}(Z=8,N=9)\) – one extra neutron very weakly bound

The Shell Model: “It was born from a thorough study of the experimental data, plotting them in different ways and looking for interconnections”

- M.G. Mayer (Nobel Lecture)
The Shell Model

Assumption: central potential and each nucleon occupies a well-defined energy level (cf. atomic electron cloud)

Schrodinger equation:

\[ H = -\frac{\hbar^2}{2m} \nabla^2 + V(r) \]

\[ H\psi = E\psi \]

Nucleons are not free particles

m, E – nucleon mass and energy

Central potential \( V(r) \) = average potential created by other nucleons

\[ H = \sum_{i=1}^{A} E_{\text{kin}}^i (r_i) + V(r_1, ..., r_2) \]

2-body interactions (simplification):

\[ H = \sum_{i=1}^{A} E_{\text{kin}}^i (r_i) + \frac{1}{2} \sum_{ij} V_{ij} (r_i, r_j) \]
The Shell Model

\[ H = \sum_{i=1}^{A} E_{\text{kin}}^i(r_i) + \frac{1}{2} \sum_{ij} V_{ij}(r_i, r_j) = \sum_{i=1}^{A} E_{\text{kin}}^i(r_i) + \sum_{i=1}^{A} U(r_i) + H_{\text{res}} \]

\[ H_{\text{res}} = \frac{1}{2} \sum_{ij} V_{ij}(r_i, r_j) - \sum_{i=1}^{A} U(r_i) \]

Shell model assumption: \( H_{\text{res}} = \text{small} \)

The shell model Hamiltonian:

\[ H_0 = \sum_{i=1}^{A} \left[ E_{\text{kin}}^i(r_i) + U(r_i) \right] \]

\[ H_0 \psi = E \psi \]

Eigenvalues \( E_1, E_2, E_3 \) (orbitals)

Nucleus wavefunction: \( \psi = \psi_1(r_1)\psi_2(r_2)\ldots\psi_A(r_A) \)

\[ E = E_1 + E_2 + \ldots + E_A \]

Note: Many of the properties of the nuclear states can be extracted from the shell model without knowledge of the wavefunction.
Quantum Harmonic Oscillator

\[ H_0 = E_{kin}(x) + V(x), \quad V(x) = \frac{1}{2} m \omega^2 x^2 \]  
\[ H_0 \psi = E \psi \implies -\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + \frac{1}{2} m \omega^2 x^2 \psi(x) = E \psi(x) \]

Solution (Quantum Mechanics)

- Wavefunction (general function)
  \[ \psi(x) = Ce^{-\alpha x^2/2}, \alpha = \frac{m \omega}{\hbar} \]
- Energy levels:
  \[ E_n = \left(n + \frac{1}{2}\right) \hbar \omega, \quad n = 0,1,2,... \]

http://hyperphysics.phy-astr.gsu.edu/hbase/quantum/hosc2.html
Quantum Harmonic Oscillator

\[ H_0 = E_{\text{kin}}(r) + V(r), \quad V(r) = \frac{1}{2} m \omega^2 r^2 \]

\[ \hbar \omega \approx 41 \text{MeV} \times A^{-1/3} \]

Note: this does not produce correct large distance behavior of the wave function but it is not important if we only analyze bound states of the nucleus

\[- \frac{\hbar^2}{2m} \nabla^2 \psi(\vec{r}) + V(r) \psi(\vec{r}) = E \psi(\vec{r})\]

Solution: \[ \psi_{nlm} = \frac{1}{r} u_{nl}(r) Y_{lm}(\theta, \varphi) \]

\[ \frac{d^2 u}{dr^2} + 2m \frac{\psi_{nl}(r)}{r} + \frac{\hbar^2}{\hbar^2} \left[ E - \frac{1}{2} m \omega^2 r^2 - \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} \right] u = 0 \]

\[ u_{nl}(r) = N_{nl} e^{-\nu r^2/2} r^{l+1} \mathcal{V}_{nl}(r) \]

\[ \nu = m \omega \hbar, \quad \mathcal{V}_{nl}(r) \quad - \text{Laguerre polynomial} \]
Quantum Harmonic Oscillator

$$\psi_{nlm} = \frac{1}{r} u_{nl}(r) Y_{lm}(\theta, \varphi)$$

$$d^2u dr^2 + \frac{2m}{\hbar^2} \left[ E - \frac{1}{2} m \omega^2 r^2 - \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} \right] u = 0$$

$$u_{nl}(r) = N_{nl} e^{-\nu r^2/2} r^{l+1} \nu_{nl}(r)$$

$$\nu = m \omega / \hbar$$

$$\nu_{nl}(r)$$ - associated Laguerre polynomial

$$\nu_{nl}(r) = L_{n+1/2}^{l+1/2} (\nu r^2) = \sum_{k=0}^{n-1} (-1)^k 2^k \binom{n-1}{k} \frac{(2l+1)!!}{(2l+2k+1)!!} (\nu r^2)^k$$

where

$$n!! = \begin{cases} 1 & n = 0, 1 \\ n(n-2)!! & n \geq 2 \end{cases}$$

$$L_{k}^{\alpha}(t)$$ are solutions of

$$t \frac{d^2 L}{dt^2} + (\alpha + 1 - t) \frac{dL}{dt} + kL = 0$$

The first few associated Laguerre polynomials:

$$L_0^{\alpha}(x) = 1$$

$$L_1^{\alpha}(x) = -x + \alpha + 1$$

$$L_2^{\alpha}(x) = \frac{1}{2} \left[ x^2 - 2(\alpha+2)x + (\alpha+1)(\alpha+2) \right]$$

......
Reminder (Quantum Mechanics)

**Quantum Harmonic Oscillator**

\[ \psi_{nlm} = \frac{1}{r} u_{nl}(r) Y_{lm}(\theta, \varphi) \]

\[ \frac{d^2 u}{dr^2} + \frac{2m}{\hbar^2} \left( E - \frac{1}{2} \omega^2 r^2 - \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} \right) u = 0 \]

\[ u_{nl}(r) = N_{nl} e^{\left(-\nu r^2/2\right)} r^{l+1} V_{nl}(r) \]

\[ \nu = \frac{m \omega}{\hbar} \]

\[ V_{nl}(r) \] - associated Laguerre polynomial

\[ V_{nl}(r) = L_{n+l-1/2}^{l+1/2}(\nu r^2) = \sum_{k=0}^{n-1} (-1)^k 2^k \binom{n-1}{k} \frac{(2l+1)!!}{(2l+2k+1)!!} (\nu r^2)^k \]

\[ \int_0^\infty u_{nl}^2(r) \, dr = 1 \Rightarrow N_{nl}^2 = \frac{2^{l-n+3}(2l+2n+1)!!}{\sqrt{\pi} (n-1)!![2(2l+1)!!]^2} \nu^{l+3/2} \]

**Normalization**

**Important:** The energy eigenvalues corresponding to \( \psi_{nlm} \):

\[ E_{nl} = \left(2n + l - \frac{1}{2}\right) \hbar \omega = \left(\Lambda + \frac{3}{2}\right) \hbar \omega = E_{\Lambda} \]

\[ n = 1, 2, 3, \ldots, \quad l = 0, 1, 2, \ldots \quad \Lambda = 2n + l - 2 \]

For each \( l \) there are \( 2(2l+1) \) states with the same energy (degenerate states). Eigenvalues which correspond to the same \( \Lambda \) are also degenerate.
### Energy Levels (harmonic oscillator)

Spectroscopic notation:
Levels are written as $nl$

$$l = 0, 1, 2, 3, 4, 5$$

$s, p, d, f, g, h$

$$\Lambda = 2n + l - 2$$

$$E_\Lambda = \hbar \omega \left( \Lambda + \frac{3}{2} \right)$$

$l = 0, \ldots, \Lambda$

- $\Lambda = odd, l = 2k + 1$
  - $e.g. \Lambda = 1 \Rightarrow l = 1$
  - $\Lambda = 3 \Rightarrow l = 1, 3$

- $\Lambda = even, l = 2k$
  - $e.g. \Lambda = 0 \Rightarrow l = 0$
  - $\Lambda = 2 \Rightarrow l = 0, 2$

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### Energy Levels (harmonic oscillator)

Spectroscopic notation:
Levels are written as \(nl\)

- \(l = 0, 1, 2, 3, 4, 5\)
- \(s, p, d, f, g, h\)
- \(n = \frac{\Lambda + 3}{2}\)

\[
E_{\Lambda} = \hbar \omega \left( \frac{\Lambda + 3}{2} \right)
\]

- \(\Lambda = 2n + l - 2\)

\[
N_{\Lambda} = (\Lambda + 1)(\Lambda + 2)
\]

\[
\sum_{\Lambda} N_{\Lambda} = \frac{1}{3}(\Lambda + 1)(\Lambda + 2)(\Lambda + 3)
\]

- \(N_{\Lambda} - \) number of protons (or neutrons)
- \(\sum_{\Lambda} N_{\Lambda} - \) accumulated number of particles for levels up to \(\Lambda\)

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<td>0</td>
<td>2</td>
<td>7/2</td>
</tr>
<tr>
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<td>1</td>
<td>3</td>
<td>9/2</td>
</tr>
<tr>
<td>2d</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>11/2</td>
</tr>
<tr>
<td>2f</td>
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<td>3</td>
<td>5</td>
<td>13/2</td>
</tr>
<tr>
<td>2g</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>15/2</td>
</tr>
<tr>
<td>2h</td>
<td>2</td>
<td>5</td>
<td>7</td>
<td>17/2</td>
</tr>
<tr>
<td>2 ....</td>
<td>2</td>
<td>...</td>
<td>....</td>
<td>.....</td>
</tr>
<tr>
<td>3s</td>
<td>3</td>
<td>0</td>
<td>4</td>
<td>11/2</td>
</tr>
<tr>
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<td>13/2</td>
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<tr>
<td>3d</td>
<td>3</td>
<td>2</td>
<td>6</td>
<td>15/2</td>
</tr>
<tr>
<td>3f</td>
<td>3</td>
<td>3</td>
<td>7</td>
<td>17/2</td>
</tr>
<tr>
<td>3 ...</td>
<td>3</td>
<td>....</td>
<td>....</td>
<td>......</td>
</tr>
</tbody>
</table>

Derivation: see Wong’s textbook
Harmonic Oscillator- Energy Levels

Magic numbers:
2, 8, 20, 28, 50, 82, 126, 184
Harmonic Oscillator - Energy Levels
**Woods-Saxon potential**

→ intermediate form between well and harmonic oscillator

\[ V_{central}(r) = \frac{-V_0}{1 + e^{(r-R)/a}} \]

- \( V_0 \approx 57 \text{ MeV} \)
- \( R \approx 1.25 A^{1/3} \text{ fm} \)
- \( a \approx 0.65 \text{ fm} \)

*Magic numbers can’t be explained by improving a form of a potential*
Elsasser had tried to explain the magic numbers by assuming that the nuclear potential in heavier nuclei is quite different from a square well. Subsequent work showed quite conclusively that a change in the shape of the potential, even a change which was quite unrealistic could not explain the magic numbers. It was kind of a jigsaw puzzle. One had many of the pieces (not only the magic number), so that one saw a picture emerging. One felt that if one had just one more piece everything would fit. The piece was found, and everything cleared up.

At that time Enrico Fermi had become interested in the magic numbers. I had the great privilege of working with him, not only at the beginning, but also later. One day as Fermi was leaving my office he asked: «Is there any indication of spin-orbit coupling?» Only if one had lived with the data as long as I could one immediately answer: «Yes, of course and that will explain everything.» Fermi was skeptical, and left me with my numerology.

---

M. Mayer (Nobel Lecture)
Woods-Saxon Potential + Spin-Orbit

\[ V(r) = V_{WS}(r) + V_{ls}(r)\left(\vec{l} \cdot \vec{s}\right) \]

\[ -\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{r}) + \left[V_{WS}(r) + V_{ls}(r)\left(\vec{l} \cdot \vec{s}\right)\right] \psi(\vec{r}) = E \psi(\vec{r}) \]

Assumption: Spin-Orbit term \sim\ small perturbation

\[ E = \langle H \rangle = \int \psi^* H \psi = E_0 + \alpha \int \psi^* V_{ls}(r)\left(\vec{l} \cdot \vec{s}\right) \psi \]

\[ \Delta E_{ls} \]

\[ \vec{l} \cdot \vec{s} = \frac{1}{2} \left(\vec{j}^2 - \vec{l}^2 - \vec{s}^2\right) \]

\[ \vec{j}^2 \psi = \hbar^2 j(j+1) \psi \]

\[ \vec{l}^2 \psi = \hbar^2 l(l+1) \psi \]

\[ \vec{s}^2 \psi = \hbar^2 s(s+1) \psi \]

\[ \int \psi^* \left(\vec{l} \cdot \vec{s}\right) \psi = \left\{ \begin{array}{ll}
\frac{l}{2} & \text{for } j = l + \frac{1}{2} \\
-\frac{1}{2}(l+1) & \text{for } j = l - \frac{1}{2}
\end{array} \right. \]

Important:

\[ \Delta E_{ls}\big|_{j=l+1/2} = -\left|\alpha\right| \langle V_{ls}(r) \rangle \frac{l}{2} \]

\[ \Delta E_{ls}\big|_{j=l-1/2} = +\left|\alpha\right| \langle V_{ls}(r) \rangle \frac{1}{2}(l+1) \]

- Experimental result: \( \alpha < 0 \)
- Lower energy for higher j!
- spin-orbit attractive
Shell Model

Levels are written as $n_l j$

$l = 0, 1, 2, 3, 4, 5$
$s, p, d, f, g, h$

- There is no restriction on the values of $l$ for a given $n$ (unlike atomic case, since nuclear potential is not Coulomb-like).

- $n$ is not the principal quantum number, it counts the number of levels with that $l$ value.

The sequence of energy levels:

<table>
<thead>
<tr>
<th>$1s_{1/2}$</th>
<th>$1p_{3/2}$</th>
<th>$1p_{1/2}$</th>
<th>$1d_{5/2}$</th>
<th>$2s_{1/2}$</th>
<th>$1d_{3/2}$</th>
<th>$1f_{7/2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>8</td>
<td>20</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The $n_l j$ level is $(2j+1)$ degenerate
Shell Model: Energy Levels

Intermediate form with spin-orbit

\[ l = 0, 1, 2, 3, 4, 5 \]
\[ s, p, d, f, g, h \]

Note: the detailed ordering of subshells above 50 varies with authors (no unique answer, since shell model is an approximation). A specific ordering must be adapted if the model is to be compared with the data.
Shell Model

Shell = group of levels of closed energy, not associated with one principal number of the oscillator
Shell Model - Applications

Shell Model can be used to predict:
the ground state spin and parity of the nuclei.

1. Angular momentum of the odd-A nuclei is determined by the angular momentum of the last nucleon in the species (proton or neutron) that is odd.

2. Even-even nuclei (Z-even and N-even; A-even) have zero ground-state spin (net angular momentum associated with even N and even Z is zero), and even parity $0^+$. 

3. In odd-odd nuclei (Z-odd and N-odd; A-even) the last neutron couples to the last proton with their intrinsic spins in parallel orientation.
Shell Model: Spin and Parity of the Nuclei in the Ground State

Odd-A nuclei (Odd-Even) Nuclei

**Extreme shell model:** total spin and parity determined by the single (unpaired) nucleon (valance or a hole)

→ There are exceptions to this rule.

E.g. Oxygen nuclei (isotope)

\[
^{17}_8 O \quad I^\pi = \frac{5^+}{2}
\]

2 closed shells for p, 2 closed shells for n + one unpaired neutron:

Notation:

p: \((1s1/2)^2(1p3/2)^4(1p1/2)^2\)

n: \((1s1/2)^2(1p3/2)^4(1p1/2)^2(1d5/2)^1\)
Shell Model: Spin and Parity of the Nuclei in the Ground State

Odd-A nuclei (Odd-Even) Nuclei

**Extreme shell model:** total spin and parity determined by the single (unpaired) nucleon (valance or a hole)

→ There are exceptions to this rule.

E.g. Fluorine nuclei

\[ ^{17}_{9}F \quad I^\pi = \frac{5}{2} \]

2 closed shells for n, 2 closed shells for p + one unpaired proton:

Notation: p: \((1s1/2)^2(1p3/2)^4(1p1/2)^2(1d5/2)^1\)

n: \((1s1/2)^2(1p3/2)^4(1p1/2)^2\)
The ground state of odd A nuclei (odd-even nuclei)

The extreme shell model: total spin and parity determined by the single (unpaired) nucleon (valence or hole).
Exercise (home): Use the shell model to predict the spins and parity of the following nuclei:

a) $^3\text{Li}^7$ (lithium)
b) $^4\text{Be}^9$ (beryllium)
c) $^{20}\text{Ca}^{40}$ (calcium)
d) $^5\text{B}^{11}$ (boron)
e) $^6\text{C}^{15}$ (carbon)
f) $^{15}\text{P}^{31}$ (Phosphorus)
g) $^{59}\text{Pr}^{141}$ (Praseodymium)
h) $^{11}\text{Na}^{23}$ (sodium)
i) $^{81}\text{Tl}^{203}$ (Thallium)
j) $^{38}\text{Sr}^{87}$ (Strontium)