Lecture 20:

4. Nuclear Force
   ▪ Low energy nucleon-nucleon scattering (pp, pn, nn)

5. Nuclei

2nd Midterm April 17 (Friday 12pm P119) Lectures 10-18
Homework3 posted on the web page (due 04/22)
Low Energy (Elastic) Scattering

Only first few terms \( f(\theta) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1)e^{i\delta_l} \sin \delta_l P_l(\cos \theta) \) are important.

If \( E < 20\text{MeV} \) then we can assume:

\[
\begin{align*}
l = 0 & \implies f(\theta) = \frac{e^{i\delta_0} \sin \delta_0}{k} \\
\sigma &= \frac{4\pi \sin^2 \delta_0}{k^2}
\end{align*}
\]

If \( E \to 0 \) the scattering amplitude is finite if \( \delta_0 \to 0 \).

The scattering length ("a") is a better parameter than a phase shift:

\[
\begin{align*}
\lim_{E \to 0} f(\theta) &= \lim_{E \to 0} \frac{\delta_0}{k} = -a \\
\lim_{E \to 0} \sigma &= \lim_{E \to 0} 4\pi \left( \frac{\sin \delta_0}{k} \right)^2 = 4\pi a^2
\end{align*}
\]

\[
k = \sqrt{\frac{2\mu E}{\hbar^2}} \quad \text{if} \quad E \to 0 \implies k \to 0
\]

\[
\begin{align*}
u_0(r) &\sim \sin(kr + \delta_0) \quad \text{if} \quad E \to 0 \implies kr + \delta_0 \approx k(r - a) \\
&= k \frac{t}{\pi} + \frac{k}{\pi/2 + \delta_i},
\end{align*}
\]

Wavefunction
The sign of the scattering length “a” shows if a resonance occurs with a negative bound state or a positive (virtual) energy.
Scattering Length – Interpretation

**Potential:**
- hard sphere of radius \( R \),
- infinitely repulsive for \( r < R \)

**Quantum Mechanics**

\[
    u_0(kr) = \begin{cases} 
    0, & r < R \\
    \sin(kr - kR), & r > R 
    \end{cases}
\]

A comparison with:

\[
    u_l(kr) \sim e^{i\delta_l} \sin(kr - l\frac{\pi}{2} + \delta_l)
\]

gives for the phase shift \( \delta_0 \) obtained for the scattering from a hard sphere:

\[
    \delta_0(E) = -kR
\]

Negative phase shift arises from a repulsive potential.

**FIGURE 7.7.** Plot of \( r A_{l=0}(r) \) versus \( r \) (with the \( e^{i\delta_0} \) factor removed). The dashed curve for \( V = 0 \) behaves like \( \sin kr \), while the solid curve is for \( S \)-wave hard-sphere scattering, shifted by \( R = -\frac{\delta_0}{k} \) from the case \( V = 0 \).
FIGURE 7.8 Plot of $u(r)$ versus $r$. (a) For $V = 0$ (dashed line). (b) For $V_0 < 0$, $\delta_0 > 0$ with the wave function (solid line) pushed in. (c) For $V_0 > 0$, $\delta_0 < 0$ with the wave function (solid line) pulled out.
Unpolarized Scattering Cross Section (p+n)

Discrepancy between the measured cross section and calculated one explained by Wigner, who proposed that the nuclear force depends on the spin.

\[ \sigma = \frac{3}{4} \sigma_t + \frac{1}{4} \sigma_s \]

Probabilities of p+n being in triplet (spins parallel) or singlet spin state (spins antiparallel)

Very low energies:

\[ \sigma = 4\pi a^2 = \frac{3}{4} \sigma_t + \frac{1}{4} \sigma_s = \pi \left( 3a_t^2 + a_s^2 \right) \]

where \( a_t \) (\( a_s \)) is the scattering length for the triplet (singlet) potentials.

\( \sigma = \pi \left( 3a_t^2 + a_s^2 \right) \) expresses incoherent combination of singlet and triplet scatterings (no interference term)
Effective Range

If \( E < 20\text{MeV} \) then we can assume \( l = 0 \)

\[
\sigma = \frac{4\pi}{k^2} \left( \sin^2 \delta_0 \right) \quad \text{- the total cross section}
\]

\[
E \rightarrow 0: \quad \lim_{E \rightarrow 0} f(\theta) = \lim_{E \rightarrow 0} \frac{\delta_0}{k} = -a
\]

\[
\sigma = 4\pi a^2
\]

The influence of the potential is represented by 1 parameter: the scattering length \( a \).

\[
E \sim \text{small}: \quad \text{(Optional: derivation see textbook)}
\]

\[
k\cot\delta = -\frac{1}{a} + \frac{1}{2}k^2r_{\text{eff}}
\]

\[
\sigma = \frac{4\pi}{k^2} \left( \frac{1}{1 + \cot^2 \delta_0} \right) \Rightarrow \sigma = \frac{4\pi a^2}{a^2k^2 + \left(1 - \frac{1}{2}ak^2r_{\text{eff}}\right)^2}
\]

The influence of the potential is represented by 2 parameters: (1) the effective range \( r_{\text{eff}} \) and (2) the scattering length \( a \).
Effective Range

\[ E \sim \text{small}: \quad \sigma = \frac{4\pi a^2}{a^2 k^2 + \left(1 - \frac{1}{2} a k^2 r_{\text{eff}}\right)^2} \]

Note: two potentials (singlet and triplet)

\[ \sigma = \frac{3}{4} \frac{4\pi a_i^2}{a_i^2 k^2 + \left(1 - \frac{1}{2} a_i^2 k^2 r_i\right)^2} + \frac{1}{4} \frac{4\pi a_s^2}{a_s^2 k^2 + \left(1 - \frac{1}{2} a_s^2 k^2 r_s\right)^2} \]

n+p results:

\[ a_i = 5.423 \pm 0.005 \text{[fm]} \]
\[ a_s = -23.71 \pm 0.001 \text{[fm]} \]
\[ r_t \sim 1.76 \text{[fm]} \]
- determined from the deuteron binding energy
\[ r_s \sim 2.56 \text{[fm]} \]
- determined from the best fit to the measured cross sections:

Textbook: section 3.8

4 parameters: \( a_i, r_i, a_s, r_s \)
Proton-Proton Scattering (Low Energies)

- repulsive Coulomb force (long range) in addition to the nuclear force
- identical particles: Pauli restrictions on the spatial and spin wavefunctions
  
  Low energies (l=0):
  spacial wave function symmetric, so spin wave function must be anti-symmetric
  Result: only singlet (spin) contributes to the cross section

- protons are not distinguishable; it implies one cannot distinguish:

The wave functions and the scattering amplitude should have contributions from \( \theta \) and \( \pi - \theta \) (interference term, purely quantum phenomenon)
Proton-Proton Scattering (Low Energies)

The scattering cross section \( l=0 \)

\[
\frac{d\sigma}{d\Omega} = \left[ \left( \frac{d\sigma}{d\Omega} \right)_c + \left( \frac{d\sigma}{d\Omega} \right)_n + \left( \frac{d\sigma}{d\Omega} \right)_{cn} \right]
\]

- Due to Coulomb potential
- Due to nuclear potential
- Interference term

Mott scattering:

\[
\left( \frac{d\sigma}{d\Omega} \right)_c = \left( \frac{e^2}{2E_p} \right)^2 \left\{ \frac{1}{\sin^4(\theta/2)} + \frac{1}{\cos^4(\theta/2)} + \frac{\cos\{\eta \ln[\tan^2(\theta/2)]\}}{\sin^2(\theta/2)\cos^2(\theta/2)} \right\}
\]

\( e^2 = 1.44\, MeVfm \)

- \( E_p \) – kinetic energy of the incident proton
- \( \theta \) – scattering angle in the center of mass system
- \( \eta = e^2/(\hbar\nu) \), \( \nu \) – relative velocity between protons

Rutherford scattering

\( \pi - \theta \) term

Interference term

B.B. Srivastava, “Fundamentals of Nuclear Physics” (solution)
Proton-Proton Scattering (Low Energies)

The scattering cross section ($l=0$)

\[
\frac{d\sigma}{d\Omega} = \left[ \left( \frac{d\sigma}{d\Omega} \right)_c + \left( \frac{d\sigma}{d\Omega} \right)_n + \left( \frac{d\sigma}{d\Omega} \right)_{cn} \right]
\]

Due to Coulomb potential

Due to nuclear potential

Interference term

Cross section due to nuclear potential:

\[
\left( \frac{d\sigma}{d\Omega} \right)_n = \frac{\sin^2 \delta_0}{k^2}
\]
Proton-Proton Scattering (Low Energies)

The scattering cross section (l=0)

\[
\frac{d\sigma}{d\Omega} = \left[ \left( \frac{d\sigma}{d\Omega} \right)_c + \left( \frac{d\sigma}{d\Omega} \right)_n + \left( \frac{d\sigma}{d\Omega} \right)_{cn} \right]
\]

Due to Coulomb potential

Due to nuclear potential

Interference term

Cross section due to interference:

\[
\left( \frac{d\sigma}{d\Omega} \right)_{cn} = -\frac{1}{2} \left( \frac{e^2}{E_p} \right)^2 \frac{\sin\delta_0}{\eta} \left\{ \cos\left[ \delta_0 + \eta \ln\sin^2(\theta/2) \right] \sin^2(\theta/2) + \cos\left[ \delta_0 + \eta \ln\cos^2(\theta/2) \right] \cos^2(\theta/2) \right\}
\]

Interference term allows to obtain the sign of \(\delta_0\)
Proton-Proton Scattering (Low Energies)

The scattering cross section ($l=0$)

$$\frac{d\sigma}{d\Omega} = \left[\left(\frac{d\sigma}{d\Omega}\right)_c + \left(\frac{d\sigma}{d\Omega}\right)_n + \left(\frac{d\sigma}{d\Omega}\right)_{cn}\right]$$

$$E_p = 3.037\,\text{MeV}$$

$$\delta_0 \sim 50.95(2)^\circ \quad \text{(best fit)}$$

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Figure 2.10: Composition of the proton-proton differential scattering cross section (solid line) by the sum of Coulomb (dotted line) and nuclear (dashed line) parts and the interference term (dashed-dotted line). The energy of the incident protons is 3.037 MeV. The experimental points were taken from [Ko66] and the best fit to (2.149) is obtained with $\delta_0 = 50.95^\circ$. Due to the singularity about 90°, only values up to this angle are shown.
Proton-Proton Scattering (Low Energies)

The scattering cross section (l=0)

$$\frac{d\sigma}{d\Omega} = \left[ \left( \frac{d\sigma}{d\Omega} \right)_c + \left( \frac{d\sigma}{d\Omega} \right)_n + \left( \frac{d\sigma}{d\Omega} \right)_{cn} \right]$$

Many measurements at different energies: $\delta_0(E)$

Attractive or repulsive nuclear potential?
Proton-Proton Scattering (Low Energies)

Effective Range

Parametrization of the cross section using 2 parameters which are connected to the potential (not depending on the details about the form of the potential)

\( a_s = -7.82 \pm 0.01 [\text{fm}] \)

\( r_s = 2.79 \pm 0.02 [\text{fm}] \)

\( a_s \sim -17 [\text{fm}] \)

\( a_t = 5.423 \pm 0.005 [\text{fm}] \)

\( a_s = -23.71 \pm 0.001 [\text{fm}] \)

\( r_t \sim 1.76 [\text{fm}] \)

\( r_s \sim 2.56 [\text{fm}] \)

\( n+p \) results:

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Negative value:
no bound state (pp) possible

Open question: is the difference between \( a_s \) from p+p and n+p real?
Neutron-Neutron Scattering (Low Energies)

Effective Range

e.g. \( n + d \rightarrow p + n + n \) (di-neutron virtual state formation)

Parametrization of the cross section using 2 parameters which are connected to the potential (not depending on the details about the form of the potential)

\[
\begin{align*}
\text{n+n results:} & \\
n + d & \rightarrow p + n + n \\
a_s &= \, -17.6 \pm 1.5 \text{[fm]} \\
r_s &= \, 3.2 \pm 1.6 \text{[fm]}
\end{align*}
\]

\[
\begin{align*}
\text{p+p results:} & \\
p + p & \rightarrow n + d + n \\
a_s &= \, -7.82 \pm 0.01 \text{[fm]} \\
r_s &= \, 2.79 \pm 0.02 \text{[fm]}
\end{align*}
\]

After subtracting the effect of Coulomb force:

\[
a_s \sim -17 \text{[fm]}
\]

Charge symmetry of the nuclear force
**Reminder**

**Composite (2 nucleons)**

**Spin:**  
\[ |\uparrow\rangle = |s = \frac{1}{2}, m_s = +\frac{1}{2}\rangle \quad |\downarrow\rangle = |s = \frac{1}{2}, m_s = -\frac{1}{2}\rangle \]

- **triplet:**  
  \[ S = 1, \ m_s = -1, 0, 1 \quad |S = 1, m_s = +1\rangle = |\uparrow\uparrow\rangle \]
  \[ |S = 1, m_s = 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \]
  \[ |S = 1, m_s = -1\rangle = |\downarrow\downarrow\rangle \]

- **singlet:**  
  \[ S = 0, \ m_s = 0 \quad |S = 0, m_s = 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \]

**Isospin:**  
\[ p = |\uparrow\rangle = |I = \frac{1}{2}, I_3 = +\frac{1}{2}\rangle \quad n = |\downarrow\rangle = |I = \frac{1}{2}, I_3 = -\frac{1}{2}\rangle \]

- **triplet:**  
  \[ I = 1, \ I_3 = -1, 0, 1 \quad |I = 1, I_3 = +1\rangle = |\uparrow\uparrow\rangle = p(1)p(2) \]
  \[ |I = 1, I_3 = 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) = \frac{1}{\sqrt{2}}(p(1)n(2) + n(1)p(2)) \]
  \[ |I = 1, I_3 = -1\rangle = |\downarrow\downarrow\rangle = n(1)n(2) \]

- **singlet:**  
  \[ I = 0, I_3 = 0 \quad |I = 0, I_3 = 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = \frac{1}{\sqrt{2}}(p(1)n(2) + n(1)p(2)) \]
Reminder

**Wave function of two nucleons (fermions)**

- **Symmetries**

\[
P_{1,2}^{\text{space}} P_{1,2}^{\text{spin}} P_{1,2}^{\text{isospin}} \psi_{j^m l}^{m_l} |I,m_I\rangle = (-1)^{I+s+l} \psi_{j^m l}^{m_l} |I,m_I\rangle \equiv P_{1,2}^{\text{tot}} \psi_{j^m l}^{m_l} |I,m_I\rangle
\]

**\(P_{1,2}^{\text{tot}}\)** - the total two-nucleon exchange operator

\[
\psi_{j^m l}^{m_l} |I,m_I\rangle \text{ is anti-symmetric } P_{1,2}^{\text{tot}} \psi(1,2) = -\psi(2,1)
\]

if \(I+s+l\) is odd
Nucleon-nucleon scattering phase shifts

Results:

- Analysis of low energy elastic scattering data

- Nuclear potential model: phase shifts calculation

See Wong’s textbook
For explanation why only certain states (combinations of I, S and I) are possible

Exercise:
In low energy isovector (isospin 1) n-p scattering, are the following states:

a) $^3P_1$   b) $^3D_1$ allowed? Explain why or why not.

Laboratory energy in MeV